Journal of Statistical Physics, Vol. 30, No. 1, 1983

Electron Mobility in Gases at Low Temperatures: The Quantum Mechanical Lorentz Gas, I

T. R. Kirkpatrick[1] and J. R. Dorfman[2]

Received June 21, 1982

The zero-field mobility of electrons in dilute gases at low temperatures is studied. We consider the case where the de Broglie wavelength of the electron is large enough that the mobility calculation must be formulated using quantum mechanical kinetic theory. The electron mobility is then computed as a function of the gas density with the approximation that the gas is considered as a collection of fixed scatterers. We evaluate the first two terms in the density expansion of the mobility, and compare these results with experiments.

KEY WORDS: Mobility; Lorentz gas; quantum kinetic theory; density expansions; binary collision operators.

1 INTRODUCTION

In two papers, of which this is the first, we examine the transport properties of electrons in gases at low temperatures where quantum mechanical effects are important. In this paper we calculate the first two terms in the density expansion of the zero-field electron mobility, μ. In the next paper, we discuss the quantum analogs of the classical divergences[(1)] in the density expansion of the electron mobility as well as the long time tail in the Green–Kubo formula for μ.[3] All of the above calculations are based on a quantum mechanical kinetic theory that is a natural generalization of classical kinetic theory.[4]

Work performed under NSF Grant CHE 77-16308.

[1] Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.

[2] Institute for Physical Science and Technology and Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742.

[3] See, for example, Ref. 2

[4] For a recent review see Ref. 3.

0022-4715/83/0100-0067$03.00/0

The theoretical and experimental interest in μ at low temperatures is due to several interesting features of its dependence on density. At the very lowest densities studied, μ has a value that is consistent with the prediction of the quantum mechanical Lorentz–Boltzmann equation. As the density is increased the mobility decreases and eventually a region of density is reached where the mobility drops several orders of magnitude over a small density range.[4] This large drop in mobility is usually attributed to electron localization. The calculations presented here are restricted to densities smaller than the localization density, and they describe the initial drop in the mobility from its low density value.

To simplify the calculations, we consider the following model of the electron–gas system:

(1) Since the mass of the electron, m, is much smaller than the mass of the gas atoms, m_g, we can treat the gas particles as though they were fixed scatterers. For a typical gas, e.g., Helium, $m/m_g \approx 10^{-3}$, which implies this should not be a serious restriction on the theory because the corrections are of order 10^{-3}.

(2) We explicitly treat the case where the interactions between the electron and the gas atoms, as well as the interactions between the gas particles themselves, are hard sphere interactions. Again, we do not believe the use of the hard sphere interaction model for the electron–atom scattering is a serious restriction on the theory for low temperatures, where our results will be applied, since under these circumstances we can use expansions in powers of the electron–atom scattering length to compute the properties of interest.[5] That is, the results of physical interest can be expressed in terms of the scattering length, which, in turn, is insensitive to the specific interparticle potential used, as long as the ratio of the scattering length a, to the thermal de Broglie wavelength of the electrons, λ, is small, and the scattering length is positive. Here the thermal wavelength of the electron is given by $\lambda = \hbar(2\pi\beta/m)^{1/2}$, where $\hbar$ is equal to Planck's constant divided by 2π, $\beta = (k_B T)^{-1}$, k_B is Boltzmann's constant, and T is the temperature. For a typical system (helium) the scattering length for the helium–electron interaction is[4] $a = 0.63 \times 10^{-8}$ cm and at $T = 4°\text{K}$, $\lambda \sim 3 \times 10^{-6}$ cm, $a/\lambda \sim 2 \times 10^{-3}$. Further, although we set up the calculation to include effects due to the interactions of the gas particles among themselves, these corrections will eventually be neglected since they are of order $n\sigma^3 \approx na^3 \approx 10^{-3}$, which is small compared to the effects calculated in this paper. Here σ is the hard core diameter of the gas particles and n is the gas density.

With the above simplifications the problem is reduced to that of calculating the electron mobility in a quantum mechanical Lorentz gas in which an electron moves in the presence of N fixed scatterers.

An important feature of the calculations given here is that we must keep track of the various expansion parameters that can appear in the theory. In classical mechanics the only expansion parameter involving the density of scatterers, n, that can appear in the calculation of μ, at zero field, is na^3 because a and $n^{-1/3}$ are the only lengths in the problem. In quantum mechanics, however, there is a new length, λ, so that the possible dimensionless expansion parameters involving the lengths $n^{-1/3}$, a, and λ are $n\lambda^3$, a/λ, $n\lambda^2 a$, $na^2\lambda$ and na^3. The first parameter, $n\lambda^3$, does not appear in our analysis since it is a parameter associated with the effects of quantum statistics that are absent in our model of one electron moving in the presence of N stationary scatterers. The second parameter, a/λ, will appear, and, as already mentioned, be taken to be much less than unity. Now, since $a/\lambda \ll 1$ the remaining dimensionless parameters satisfy the inequality[5] $n\lambda^2 a \gg na^2\lambda \gg na^3$. In the course of our calculations we will show that for low temperatures the parameter $n\lambda^2 a$ does not appear in the calculation of the density expansion of μ so that the most important expansion parameter is $na^2\lambda$. It should be remarked that the appearance of the parameter $na^2\lambda$ can be understood from the fact that it is simply the ratio of the de Broglie wavelength of the electron to its mean free path in the system of scatterers, λ/l, where $l \approx (na^2)^{-1}$ is the mean free path of the electron. That is, the first density correction to the dilute gas result for μ becomes important when the thermal wavelength is comparable to the mean free path of the moving particle. We also note that in typical experiments $na^2\lambda \approx 10^{-1}$, and that $n\lambda^2 a \approx 50$.

The Green–Kubo formula for the mobility of an electron moving in the presence of fixed scatterers is[6]

$$\mu = \frac{i}{\hbar}\frac{e}{m}\lim_{\epsilon\to 0}\lim_{\substack{N\to\infty \\ \Omega\to\infty \\ N/\Omega = n}}\int_0^\infty dt\, e^{-\epsilon t}\int \frac{d\mathbf{R}^N}{\Omega^{N+1}}\,\mathrm{Tr}\left[\hat{z}, \hat{\rho}_{\mathrm{eq}}(\hat{\mathbf{x}}, \mathbf{R}^N)\right]\hat{p}_z(t) \quad (1.1)$$

where $\hat{\mathbf{x}} = \hat{\mathbf{r}}, \hat{\mathbf{p}}$ are the quantum mechanical operators for the electron's position and momentum, e is the electron charge, the $\mathbf{R}^N = (\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_N)$ are the positions of the N fixed scatterers in a volume Ω, [,] denotes commutator, Tr indicates a trace over a complete set of states for the electron, $\hat{\rho}_{\mathrm{eq}}(\hat{\mathbf{x}}, \mathbf{R}^N)$ is the statistical density operator in the canonical ensemble (cf. Section 2), and $\hat{p}_z(t)$ is the z component of the momentum operator of the electron at time t. Throughout this paper we use a caret to denote a quantum mechanical operator.

[5] In typical experiments[4] $a/\lambda \sim 2\times 10^{-3}$, $n\lambda^2 a \sim 50$, $na^2\lambda \sim 0.1$–1 and $na^3 \sim 10^{-3}$.

[6] See, for example, Ref. 6.

The purpose of this paper is to evaluate Eq. (1.1) for a system where the reduced density of scatterers, $na^2\lambda \approx \lambda/l$, is small. We will show that the first two terms in a density expansion of μ are (when $a/\lambda \ll 1$)

$$\mu = \frac{\beta e}{3(2\pi\beta m)^{1/2} n\pi a^2}\left\{1 - \pi^2 na^2\lambda + O\left[(na^2\lambda)^2, na^3\right]\right\}$$

$$\equiv \mu_B\left[1 - \pi^2 na^2\lambda\right] \tag{1.2}$$

with μ_B the mobility given by the quantum mechanical Boltzmann equation.

The outline of this paper is as follows. In Section 2 we give the basic formulas needed for the calculation of μ as well as a formal density expansion of μ in terms of dynamical time displacement operators and reduced density operators. This is accomplished by generalizing a procedure due to Zwanzig[7] for obtaining virial expansions of transport coefficients for classical systems to quantum systems. As a result, we obtain a kinetic equation for the Laplace transform of a time correlation function which, when solved, gives μ as a density expansion in terms of dynamical operators and reduced density matrices. In Section 3 the reduced density matrices are computed by using the equilibrium binary collision expansion (BCE) of Lee and Yang.[8] In Section 4 we use the results of Sections 2 and 3 to derive the first term in the density expansion of μ, the Boltzmann contribution, μ_B, in Eq. (1.2), and in Section 5 we use a nonequilibrium binary collision expansion[7] to compute the first density correction to μ_B [cf. Eq. (1.2)]. We then compare Eq. (1.2) with existing experimental data for μ. In Section 6 we discuss the relationship between our results and that of previous workers in the field and make some concluding remarks.

2. BASIC FORMULAS AND CLUSTER EXPANSION OF THE ELECTRON MOBILITY

We begin by considering the equilibrium density operator in Eq. (1.1) given by

$$\hat{\rho}_{eq}(\hat{\mathbf{x}}, \mathbf{R}^N) = \frac{\exp\left[-\beta\hat{H}(\hat{\mathbf{x}}, \mathbf{R}^N)\right]}{Z_{N+1}(\beta, \Omega)} \tag{2.1}$$

where Z_{N+1} is the canonical partition function for a system consisting of N fixed scatterers and a moving electron of mass m,

$$Z_{N+1}(\beta, \Omega) = \int \frac{d\mathbf{R}^N}{\Omega^{N+1}} \operatorname{Tr} \exp\left[-\beta\hat{H}(\hat{\mathbf{x}}, \mathbf{R}^N)\right] \tag{2.2}$$

where $\hat{H}(\hat{\mathbf{x}}, \mathbf{R}^N)$ is the Hamiltonian operator. $\hat{H}(\hat{\mathbf{x}}, \mathbf{R}^N)$ can be conveniently written

$$\hat{H}(\hat{\mathbf{x}}, \mathbf{R}^N) = \hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}^N) + H_{\text{st}}(\mathbf{R}^N) \tag{2.3}$$

with

$$\hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}^N) = \hat{H}_0(\tilde{\mathbf{p}}) + \sum_{i=1}^{N} \hat{V}_d(\hat{\mathbf{r}} - \mathbf{R}_i) \tag{2.4a}$$

$$\hat{H}_p(\hat{\mathbf{p}}) = \frac{\hat{p}^2}{2m} \tag{2.4b}$$

$$H_{\text{st}}(\mathbf{R}^N) = \frac{1}{2} \sum_{i \neq j}^{N} V_{\text{st}}(\mathbf{R}_i - \mathbf{R}_j) \tag{2.4c}$$

where $\hat{V}_d(\hat{\mathbf{r}} - \mathbf{R}_i)$ is the dynamical interaction potential between the moving electron and the ith scatterer. In the coordinate representation:

$$\begin{aligned} V_d(\mathbf{r} - \mathbf{R}_i) &= \infty \qquad \text{for} \quad |\mathbf{r} - \mathbf{R}_i| \leqslant a \\ V_d(\mathbf{r} - \mathbf{R}_i) &= 0 \qquad \text{for} \quad |\mathbf{r} - \mathbf{R}_i| > a \end{aligned} \tag{2.5}$$

Similarly, $V_{\text{st}}(\mathbf{R}_i - \mathbf{R}_j)$ is the interaction potential between two stationary scatterers i and j:

$$\begin{aligned} V_{\text{st}}(\mathbf{R}_i - \mathbf{R}_j) &= \infty \qquad \text{for} \quad |\mathbf{R}_i - \mathbf{R}_j| \leqslant \sigma \\ V_{\text{st}}(\mathbf{R}_i - \mathbf{R}_j) &= 0 \qquad \text{for} \quad |\mathbf{R}_i - \mathbf{R}_j| > \sigma \end{aligned} \tag{2.6}$$

with σ the diameter of the scatterers. Also, in Eq. (1.1), $\hat{p}_z(t)$ is given by

$$\begin{aligned} \hat{p}_z(t) &= \exp\left[i \frac{\hat{H}}{\hbar} (\hat{\mathbf{x}}, \mathbf{R}^n) t \right] \hat{p}_z \exp\left[-i \frac{\hat{H}}{\hbar} (\hat{\mathbf{x}}, \mathbf{R}^N) t \right] \\ &= \exp\left[i \frac{\hat{H}_d}{\hbar} (\hat{\mathbf{x}}, \mathbf{R}^N) t \right] \hat{p}_z \exp\left[-i \frac{\hat{H}_d}{\hbar} (\hat{\mathbf{x}}, \mathbf{R}^N) t \right] \\ &\equiv \exp\left[i \mathfrak{L}(\hat{\mathbf{x}}, \mathbf{R}^n) t \right] \hat{p}_z \end{aligned} \tag{2.7}$$

where we have used the fact that $H_{\text{st}}(\mathbf{R}^N)$ commutes with $\hat{p}_z$ and the last equality in Eq. (2.7) defines the Liouville operator, $\mathfrak{L}(\hat{\mathbf{x}}, \mathbf{R}^N) = (1/\hbar)[\hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}^N), \ ]$ for an electron moving through an array of N fixed scatterers. It should be remarked that $\exp[i\mathfrak{L}(\hat{\mathbf{x}}, \mathbf{R}^N)t]$ is a superoperator that acts on operators and not state functions.

In order to facilitate the evaluation of Eq. (1.1) for μ as a power series in the density of scatterers we make an operator cluster expansion of the quantum mechanical time displacement operator $\exp[it\mathfrak{L}(\hat{\mathbf{x}}, \mathbf{R}^N)]$. For this simple system the Ursell cluster expansion of the operator $\exp[it\mathfrak{L}(\hat{\mathbf{x}}, \mathbf{R}^N)]$ is

given by

$$\exp\left[it\mathfrak{L}(\hat{\mathbf{x}},\mathbf{R}^N)\right] = \exp\left[it\mathfrak{L}(\hat{\mathbf{p}})\right] + \sum_{i=1}^{N}\hat{\mathfrak{U}}_1(\hat{\mathbf{x}},\mathbf{R}_i,t) + \frac{1}{2!}\sum_{i\neq j}^{N}\hat{\mathfrak{U}}_2(\hat{\mathbf{x}},\mathbf{R}_i\mathbf{R}_j,t) + \cdots + \hat{\mathfrak{U}}_n(\hat{\mathbf{x}},\mathbf{R}_1,\ldots,\mathbf{R}_N,t) \tag{2.8}$$

where $\mathfrak{L}(\hat{\mathbf{p}}) = (1/\hbar)[\hat{H}_0(\hat{\mathbf{p}}),\ ]$ and the Ursell operators, $\hat{\mathfrak{U}}_l$, can be obtained from Eq. (2.8) by considering the cases $N = 1, 2, \ldots,$ successively. This leads to the following results for the operators $\hat{\mathfrak{U}}_1, \hat{\mathfrak{U}}_2, \ldots$:

$$\hat{\mathfrak{U}}_1(\hat{\mathbf{x}},\mathbf{R}_1,t) = \exp\left[it\mathfrak{L}(\hat{\mathbf{x}},\mathbf{R}_1)\right] - \exp\left[it\mathfrak{L}(\hat{\mathbf{p}})\right] \tag{2.9a}$$

$$\hat{\mathfrak{U}}_2(\hat{\mathbf{x}},\mathbf{R}_1\mathbf{R}_2,t) = \exp\left[it\mathfrak{L}(\hat{\mathbf{x}},\mathbf{R}_1\mathbf{R}_2)\right] - \exp\left[it\mathfrak{L}(\hat{\mathbf{x}},\mathbf{R}_1)\right] - \exp\left[it\mathfrak{L}(\hat{\mathbf{x}},\mathbf{R}_2)\right] + \exp\left[it\mathfrak{L}(\hat{\mathbf{p}})\right] \tag{2.9b}$$

After inserting Eqs. (2.9) into Eq. (1.1) one can show that the electron mobility is given by

$$\mu = \frac{i}{\hbar}\frac{e}{m}\lim_{\epsilon\to 0}\lim_{\substack{N,\Omega\to\infty \\ N/\Omega = n}}\frac{\mathrm{Tr}}{\Omega}\left[\hat{z},\hat{g}(\hat{p})\right]\hat{\Phi}(\hat{\mathbf{p}},\epsilon) \tag{2.10}$$

where $\hat{\Phi}(\hat{\mathbf{p}},\epsilon)$ is defined by

$$\hat{\Phi}(\hat{\mathbf{p}},\epsilon) = \hat{\mathcal{G}}_0(\hat{\mathbf{p}})\hat{p}_z + \sum_{l=1}^{N}\frac{n^l}{l!}\int d\mathbf{R}^l\left[\hat{z},\hat{g}(\hat{\mathbf{p}})\right]^{-1}\left[\hat{z},\hat{g}(\hat{\mathbf{x}},\mathbf{R}^l)\right]\hat{\mathfrak{U}}_l(\hat{\mathbf{x}},\mathbf{R}^l,\epsilon)\hat{p}_z \tag{2.11}$$

Here $\hat{\mathcal{G}}_0(\hat{\mathbf{p}}) = [\epsilon - i\mathfrak{L}(\hat{\mathbf{p}})]^{-1}$ is the free particle propagator, $\hat{\mathfrak{U}}_l(\hat{\mathbf{x}},\mathbf{R}^l,\epsilon)$ is the Laplace transform of $\hat{\mathfrak{U}}_l(\hat{\mathbf{x}},\mathbf{R}^l,t)$, and the $\hat{g}(\hat{\mathbf{x}},\mathbf{R}^l)$ are reduced density operators defined by

$$\frac{\hat{g}(\hat{\mathbf{p}})}{\Omega} = \int\frac{d\mathbf{R}^N}{\Omega^{N+1}}\frac{\exp\left[-\beta\hat{H}(\hat{\mathbf{x}},\mathbf{R}^N)\right]}{Z_{N+1}(\beta,\Omega)} \tag{2.12a}$$

$$\frac{n\hat{g}(\hat{\mathbf{x}},\mathbf{R}_1)}{\Omega} = N\int\frac{d\mathbf{R}^{N-1}}{\Omega^{N+1}}\frac{\exp\left[-\beta\hat{H}(\hat{\mathbf{x}},\mathbf{R}^N)\right]}{Z_{N+1}(\beta,\Omega)} \tag{2.12b}$$

$$n^2\hat{g}(\hat{\mathbf{x}},\mathbf{R}_1,\mathbf{R}_2) = N(N-1)\int\frac{d\mathbf{R}^{N-2}}{\Omega^{N+1}}\frac{\exp\left[-\beta\hat{H}(\hat{\mathbf{x}},\mathbf{R}^N)\right]}{Z_{N+1}(\beta,\Omega)} \tag{2.12c}$$

$$\vdots$$

The factors of Ω in Eqs. (2.12) have been included so that the reduced density operators remain finite in the thermodynamic limit ($N\to\infty$, $\Omega\to\infty$, $N/\Omega\to n$; cf. Section 3). In giving Eq. (2.11) we have also formally defined the inverse of $[\hat{z},\hat{g}(\hat{\mathbf{p}})]$ to be $[\hat{z},\hat{g}(\hat{\mathbf{p}})]^{-1}$. In Section 3 we will show how these reduced density operators, and $[\hat{z},\hat{g}(\hat{\mathbf{p}})]^{-1}$, can be computed as a

power series in the density. We note that because the reduced density operators themselves have a density expansion, Eqs. (2.10) and (2.11) do not yet constitute a density expansion for μ.

To proceed further, we first point out that each of the terms in the expansion of μ given by Eqs. (2.10) and (2.11) does not exist in the limit $\epsilon \to 0$. To see this we need only note that $\mathcal{G}_0(\hat{\mathbf{p}})\hat{p}_z = \epsilon^{-1}\hat{p}_z$, since $\hat{p}_z$ commutes with $\mathcal{L}(\hat{p})$, so that the first term in Eq. (2.11) diverges as $\epsilon \to 0$. Further, using the nonequilibrium binary collision expansion of the $\hat{\mathcal{U}}_l$'s given in Section 5, one can convince oneself that each $\hat{\mathcal{U}}_l(\epsilon)$ diverges as $\epsilon^{-(l+1)}$ as $\epsilon \to 0$.[7] These divergence difficulties are also present in the classical version of this calculation and they are eliminated here, as in the classical case, by rearranging the density expansion of $\hat{\Phi}(\hat{\mathbf{p}}, \epsilon)$, given by Eq. (2.11), using a procedure due to Zwanzig.[7]

Zwanzig's method is to define a new set of superoperators $\hat{\mathcal{B}}_l(\hat{\mathbf{x}}, \mathbf{R}^l, \epsilon)$ by writing Eq. (2.11) identically as

$$\hat{\Phi}(\hat{\mathbf{p}}, \epsilon) = \left[\epsilon - i\mathcal{L}(\hat{\mathbf{p}}) - \sum_{l=1}^{N} \frac{n^l}{l!} \int d\mathbf{R}^l \hat{\mathcal{B}}_l(\hat{\mathbf{x}}, \mathbf{R}^l, \epsilon)\right]^{-1} \hat{p}_z \tag{2.13a}$$

or

$$\left[\epsilon - \sum_{l=1}^{N} \frac{n^l}{l!} \int d\mathbf{R}^l \hat{\mathcal{B}}_l(\hat{\mathbf{x}}, \mathbf{R}^l, \epsilon)\right] \hat{\Phi}(\hat{\mathbf{p}}, \epsilon) = \hat{p}_z \tag{2.13b}$$

where we have used that $\mathcal{L}(\hat{\mathbf{p}})\hat{\Phi}(\hat{\mathbf{p}}, \epsilon) = 0$, since $\hat{\Phi}(\hat{\mathbf{p}}, \epsilon)$ is diagonal in the momentum representation as can be easily shown. By expanding the right-hand side of Eq. (2.13a) in powers of n and equating the coefficients to those of the identical powers of n on the right-hand side of Eq. (2.11), we obtain

$$\int d\mathbf{R}_1 \hat{\mathcal{B}}_1(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon)$$

$$= \int d\mathbf{R}_1 \mathcal{G}_0^{-1}(\hat{\mathbf{p}})[\hat{z}, \hat{g}(\hat{\mathbf{p}})]^{-1}[\hat{z}, \hat{g}(\hat{\mathbf{x}}, \mathbf{R}_1)]\hat{\mathcal{U}}_1(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon)\mathcal{G}_0^{-1}(\hat{\mathbf{p}}) \tag{2.14a}$$

$$\int d\mathbf{R}_1 \int d\mathbf{R}_2 \hat{\mathcal{B}}_1(\hat{\mathbf{x}}, \mathbf{R}_1, \mathbf{R}_2, \epsilon)$$

$$= \int d\mathbf{R}_1 \int d\mathbf{R}_2 \mathcal{G}_0^{-1}(\hat{\mathbf{p}})[\hat{z}, \hat{g}(\hat{\mathbf{p}})]^{-1}[\hat{z}, \hat{g}(\hat{\mathbf{x}}, \mathbf{R}_1\mathbf{R}_2)]\hat{\mathcal{U}}_2(\hat{\mathbf{x}}, \mathbf{R}_1\mathbf{R}_2, \epsilon)\mathcal{G}_0^{-1}(\hat{\mathbf{p}})$$

$$-2! \int d\mathbf{R}_1 \int d\mathbf{R}_2 \hat{\mathcal{B}}_1(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon)\mathcal{G}_0(\hat{\mathbf{p}})\hat{\mathcal{B}}_1(\hat{\mathbf{x}}, \mathbf{R}_2, \epsilon) \tag{2.14b}$$

and so on.

In the following sections we will explicitly show that the above procedure allows one to compute the lowest-order contribution to μ, due to

[7] The appearance of these divergencies is reasonable since it is known that the low density μ is of $O(n^{-1})$. From this it follows that an expansion of μ in positive powers of n is not valid.

$\hat{\mathfrak{B}}_1$, as well as the first density correction to this value, due to $\hat{\mathfrak{B}}_2$. Further, in the next paper of this series we will show that if one proceeds to the next order in n, and considers $\hat{\mathfrak{B}}_3$, then one finds logarithmically divergent terms of relative order $(na^2\lambda)^2 \log \epsilon$.

3. THE REDUCED DENSITY MATRICES IN EQUILIBRIUM

In the previous section we derived a formally exact operator kinetic equation for $\hat{\Phi}(\hat{\mathbf{p}}, \epsilon)$ where both dynamical operators, $\hat{\mathfrak{A}}_l$, as well as commutators of the form $[\hat{z}, \hat{g}(\hat{\mathbf{x}}, \mathbf{R}^l)]$ appear, with $\hat{g}(\hat{\mathbf{x}}, \mathbf{R}^l)$ the reduced density operators defined by Eqs. (2.12). In this section we show how to calculate the $\hat{g}(\hat{\mathbf{x}}, \mathbf{R}^l)$ as a power series in the density. Further, we show that if $a/\lambda \ll 1$ then the largest dimensionless density expansion parameter that appears in these expansions is $na^2\lambda$, i.e., the parameter $n\lambda^2 a$ does not appear.

As a simplifying approximation we will replace $\hat{H}(\hat{\mathbf{x}}, \mathbf{R}^N)$ in Eqs. (2.12) by $\hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}^N)$ given by Eq. (2.4a). That is, we neglect the interactions between the N scatterers which is equivalent to neglecting excluded volume corrections of $O(na^3)$. Since the leading density corrections calculated here are of $O(na^2\lambda)$ this is justified when $a/\lambda \ll 1$. Using this approximation we will compute the reduced density matrices given by Eqs. (2.12) by making an operator Ursell cluster expansion of $\exp[-\beta \hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}^N)]$ and by evaluating the cluster functions by means of an equilibrium binary collision expansion similar to that given by Lee and Yang.[8] When carrying out this procedure, we will find that for low temperatures, the most convenient form for the density expansion of the equilibrium distribution functions is obtained by using an inversion procedure similar to that used to define the $\hat{\mathfrak{B}}_l$ in Section 2. In this way we will show that the terms of $O(n\lambda^2 a)$ in $Z_{N+1}(\beta, \Omega)$ are canceled by those in $\int d\mathbf{R}^{N-l} \exp[-\beta \hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}^N)]$ [cf. Eq. (2.12)].

We first calculate the partition function $Z_{N+1}(\beta, \Omega)$. Here we use the operator cluster expansion of $\exp[-\beta \hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}^N)]$ defined in analogy with Eq. (2.8) by

$$\exp\left[-\beta \hat{H}_d(\hat{\mathbf{x}}, \hat{\mathbf{R}}^N)\right] = \exp\left[-\beta \hat{H}_0(\hat{\mathbf{p}})\right] + \sum_{i=1}^{N} \hat{u}_1(\hat{\mathbf{x}}, \mathbf{R}_i, \beta) + \frac{1}{2!} \sum_{i \neq j}^{N} \hat{u}_2(\hat{\mathbf{x}}, \mathbf{R}_i, \mathbf{R}_j, \beta) + \cdots + \hat{u}_N(\hat{\mathbf{x}}, \mathbf{R}_1, \mathbf{R}_2 \ldots \mathbf{R}_N, \beta) \quad (3.1)$$

Using Eq. (3.1) for $N = 1, 2, \ldots,$ one obtains the explicit form of $\hat{u}_l(\hat{\mathbf{x}}, \mathbf{R}^l, \beta)$, successively:

$$\hat{u}_1(\hat{\mathbf{x}}, \mathbf{R}_1, \beta) = \exp\left[-\beta \hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}_1)\right] - \exp\left[-\beta \hat{H}_0(\hat{\mathbf{p}})\right] \tag{3.2a}$$

$$\hat{u}_2(\hat{\mathbf{x}}, \mathbf{R}_1, \mathbf{R}_2, \beta) = \exp\left[-\beta \hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}_1, \mathbf{R}_2)\right] - \exp\left[-\hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}_1)\right] - \exp\left[-\beta \hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}_2)\right] + \exp\left[-\beta \hat{H}_0(\hat{\mathbf{p}})\right] \tag{3.2b}$$

$$\vdots$$

Inserting Eq. (3.1) into Eq. (2.2) we obtain in the thermodynamic limit

$$Z(\beta, n) = \lim_{\substack{N,\Omega \to \infty \\ N/\Omega \to n}} Z_{N+1}(\beta, \Omega)$$

$$= \frac{\mathrm{Tr}}{\Omega}\left[\exp\left[-\beta \hat{H}_0(\hat{\mathbf{p}})\right] + \sum_{l=1}^{\infty} \frac{n^l}{l!} \int d\mathbf{R}^l \hat{u}_l(\hat{\mathbf{x}}, \mathbf{R}^l, \beta)\right] \tag{3.3}$$

To evaluate $Z(\beta, n)$ it is convenient to represent the operators in Eq. (3.3) in terms of their Laplace transforms with respect to β. That is, we write Eq. (3.3) as

$$Z(\beta, n) = \frac{\mathrm{Tr}}{\Omega} \frac{1}{(2\pi i)} \int_{\gamma - i\infty}^{\gamma + i\infty} ds\, e^{\beta s}\left[\frac{1}{s + \hat{H}_0(\hat{\mathbf{p}})} + \sum_{l=1}^{\infty} \frac{n^l}{l!} \int d\mathbf{R}^l \hat{u}_l(\hat{\mathbf{x}}, \mathbf{R}^l, s)\right] \tag{3.4}$$

with $\hat{u}_l(\hat{\mathbf{x}}, \mathbf{R}^l, s)$ the Laplace transform of $\hat{u}_l(\hat{\mathbf{x}}, \mathbf{R}^l, \beta)$.

The structure of the density expansion of $Z(\beta, n)$ given by Eq. (3.4) can be determined by introducing a binary collision expansion of the $\hat{u}_l(\hat{\mathbf{x}}, \mathbf{R}^l, s)$. The binary collision operator $\hat{t}$, that describes the interaction between the electron and a scatterer at $\mathbf{R}_1$ is defined by[5]

$$\hat{t}(\hat{\mathbf{x}}, \mathbf{R}_1, s) = \hat{V}_d(\hat{\mathbf{r}} - \mathbf{R}_1) \frac{1}{\left[s + \hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}_1)\right]}\left[s + \hat{H}_0(\hat{\mathbf{p}})\right] \tag{3.5a}$$

The matrix elements of $\hat{t}$ in the momentum representation of the moving particle are

$$\langle \mathbf{p}|\hat{t}(\hat{\mathbf{x}}, \mathbf{R}_1, s)|\mathbf{p}_1\rangle = t_1(\mathbf{p}, \mathbf{p}_1, s; \mathbf{R}_1) \tag{3.5b}$$

with $|\mathbf{p}\rangle$, the free particle momentum eigenstate,

$$\langle \mathbf{r}|\mathbf{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp(i\mathbf{p} \cdot \mathbf{r}/\hbar) \tag{3.5c}$$

We remark that the t matrix given by Eq. (3.5b) is directly related to the t matrix that appears in quantum mechanical scattering theory.[5] Equation

(3.4) can now be written as

$$Z(\beta,n) = \frac{1}{(2\pi\hbar)^3}\int d\mathbf{p}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{\beta s} \times\left[G_0(p) + \sum_{l=1}^{\infty}\frac{n^l}{l!}\int d\mathbf{R}^l u_l(\mathbf{p},\mathbf{R}^l,s)\right] \tag{3.6}$$

with $G_0(p) = [s + p^2/2m]^{-1}$ the "equilibrium" free particle propagator. In Eq. (3.6) the first few u_l's are given in terms of t matrices by

$$\int d\mathbf{R}_1\, u_1(\mathbf{p},\mathbf{R}_1,s) = -\int d\mathbf{R}_1\, G_0(p)\hat{t}_1(\mathbf{p})G_0(p) \tag{3.7a}$$

and

$$\frac{1}{2!}\int d\mathbf{R}_1\int d\mathbf{R}_2\, u_2(\mathbf{p},\mathbf{R}_1,\mathbf{R}_2,s) = \int d\mathbf{R}_1\int d\mathbf{R}_2\,\{G_0(p)\hat{t}_1(\mathbf{p})G_0(p)\hat{t}_2(\mathbf{p})G_0(\mathbf{p}) - G_0(p)\hat{t}_1(\mathbf{p})G_0(\mathbf{p})\hat{t}_2(\mathbf{p})G_0(\mathbf{p})\hat{t}_1(\mathbf{p})G_0(\mathbf{p}) + \cdots\} \tag{3.7b}$$

where the sign of each term is determined by a factor $(-1)^j$ with j the number of $\hat{t}$ operators and

$$\hat{t}_i(\mathbf{p}) = \int d\mathbf{p}'\, t_i(\mathbf{p},\mathbf{p}',s;\mathbf{R}_i)P(\mathbf{p}',\mathbf{p}) \tag{3.8}$$

where $P(\mathbf{p}',\mathbf{p})$ is a permutation operator that changes the index $\mathbf{p}$ to $\mathbf{p}'$ in any function $f(\mathbf{p})$ on which it acts.

The "naive" density expansion, Eq. (3.6), for $Z(\beta,n)$ can only be used to compute the partition function for sufficiently high temperatures and low densities such that $n\lambda^2 a \ll 1$. To see this, truncate Eq. (3.6) by keeping only the term of $O(n)$. If one then uses the identity (proved in Appendix A):

$$t_1(\mathbf{p},\mathbf{p}',s;\mathbf{R}_1) = \exp\left[-i\frac{\mathbf{R}_1}{\hbar}\cdot(\mathbf{p}-\mathbf{p}')\right]t(\mathbf{p},\mathbf{p}',s) \tag{3.9}$$

where $t(\mathbf{p},\mathbf{p}',s)$ is independent of $\mathbf{R}_1$, in Eq. (3.6) one finds

$$Z(\beta,n) = \frac{1}{(2\pi\hbar)^3}\int d\mathbf{p}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{\beta s} \times\left[G_0(p) - n(2\pi\hbar)^3 t(\mathbf{p},\mathbf{p},s)G_0^2(p)\right] \tag{3.10}$$

In Appendix A we show that to $O(a)$, $t(\mathbf{p},\mathbf{p},s)$ is given by $a/4\pi^2\hbar m$. Here a can be either the range of the hard-core potential defined by Eq. (2.5), or

the low-energy, s-wave scattering length for a more general potential. Using this in Eq. (3.10) we easily obtain

$$Z(\beta,n) = \frac{1}{\lambda^3}\left[1 - n\lambda^2 a\right] + O(a^2) \tag{3.11}$$

Further, one can easily convince oneself that in $Z(\beta,n)$ all powers of $n\lambda^2 a$ appear. Because we are interested in density and temperature regimes where $n\lambda^2 a$ is not small (see note 5 above) all terms of this order must be resummed.

As mentioned above, this resummation is most easily carried out by using the same rearrangement of Eq. (3.6) as was given in Section 2 for Eq. (2.11). To this end we define a new set of operators, $\hat{b}_l(\mathbf{p},\mathbf{R}^l,s)$, by means of the identity

$$g_0(p) + \sum_{l=1}^{\infty} \frac{n^l}{l!} \int d\mathbf{R}^l u_l(\mathbf{p},\mathbf{R}^l,s) = \left[s + \frac{p^2}{2m} + \sum_{l=1}^{\infty} n^l \int d\mathbf{R}^l \hat{b}_l(\mathbf{p},\mathbf{R}^l,s) \right]^{-1} \tag{3.12}$$

As in Section 2, we expand the right-hand side in the powers of the density and by equating powers of n, we can determine the $\hat{b}$'s. Using Eq. (3.7), we find that the first few $\hat{b}$'s are given in terms of t operators as

$$\int d\mathbf{R}_1 \hat{b}_1(\mathbf{p},\mathbf{R}_1;s) = \int d\mathbf{R}_1 \hat{t}_1(\mathbf{p}) \tag{3.13a}$$

$$\begin{aligned} &\int d\mathbf{R}_1 \int d\mathbf{R}_2 \hat{b}_2(p,\mathbf{R}_1\mathbf{R}_2;s) \\ &\quad = \int d\mathbf{R}_1 \int d\mathbf{R}_2 \big[\hat{t}_1(\mathbf{p}) G_0(p) \hat{t}_2(\mathbf{p}) G_0(p) \hat{t}_1(\mathbf{p}) \\ &\qquad - \hat{t}_1(\mathbf{p}) G_0(p) \hat{t}_2(\mathbf{p}) G_0(p) \hat{t}_1(\mathbf{p}) G_0(\mathbf{p}) \hat{t}_2(\mathbf{p}) + \cdots \big] \end{aligned} \tag{3.13b}$$

The insertion of Eq. (3.12) in Eq. (3.6) yields

$$\begin{aligned} Z(\beta,n) = \frac{1}{(2\pi\hbar)^3} \int d\mathbf{p} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} ds\, e^{\beta s} \\ \times \left[s + \frac{p^2}{2m} + \sum_{l=1}^{\infty} n^l \int d\mathbf{R}^l \hat{b}_l(\mathbf{p},\mathbf{R}^l;s) \right]^{-1} \end{aligned} \tag{3.14}$$

We can now determine $Z(\beta,n)$ where all powers of $n\lambda^2 a$ are resummed. If we denote the lowest-order approximation, by $Z_1(\beta,n)$ where only $\hat{b}_1(p,\mathbf{R}_1,s)$ is retained and use Eqs. (3.8), (3.9), and (3.13a), we find

that

$$Z_1(\beta,n) = \frac{1}{(2\pi\hbar)^3}\int d\mathbf{p}\,\frac{1}{(2\pi i)}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{\beta s} \times\left[s + \frac{p^2}{2m} + n(2\pi\hbar)^3 t(\mathbf{p},\mathbf{p},s)\right]^{-1} \tag{3.15}$$

In Appendix A [cf. Eq. (A.8)] we show that to $O(a^2)$ $t(\mathbf{p},\mathbf{p},s)$ is given by

$$t(\mathbf{p},\mathbf{p}_1,s) = \frac{a}{4\pi^2\hbar m}\left[1 + \frac{a}{\hbar}(2ms)^{1/2}\right] + O(a^3) \tag{3.16}$$

Inserting Eq. (3.16) into Eq. (3.15), keeping the contribution of $O(a)$ only, we obtain

$$Z_1(\beta,n) = \frac{e^{-\beta E_0}}{\lambda^3} = \frac{e^{-n\lambda^2 a}}{\lambda^3} \tag{3.17a}$$

where

$$E_0 = na\frac{\hbar^2 2\pi}{m} \tag{3.17b}$$

In the next part of this section we show that, for low temperatures, one cannot consistently retain the term of $O(a^2)$ in Eq. (3.15) without performing further infinite resummations. Further, we will show that each approximation to $Z(\beta,n)$ is proportional to $e^{-\beta E_0}$ and that each term in the numerator of Eq. (2.12) is also proportional to $e^{-\beta E_0}$ so that all terms of $O(n\lambda^2 a)$ in the reduced density matrices eventually cancel.

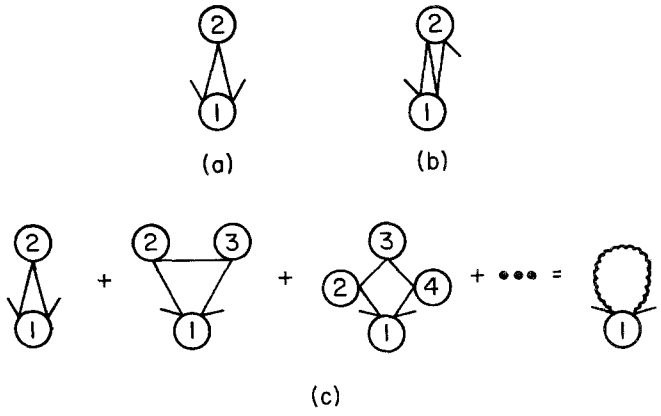

Fig. 1. (a) The equilibrium two-scatterer ring diagram. (b) The equilibrium two-scatterer rattling ring diagram. (c) The equilibrium two-, three-, . . . , scatterer ring diagrams and their sum. The wavy line here represents renormalized propagation.

The first density corrections to Eq. (3.15) are obtained by including $\hat{b}_2(\mathbf{p},\mathbf{R}_1,\mathbf{R}_2,s)$, given by Eq. (3.13b), in Eq. (3.14). To catalog the various terms that appear in the analysis we use diagrammatic technique. The first term on the right-hand side of Eq. (3.13b) will be called the two-scatterer ring diagram due to its graphical representation given in Fig. 1a, where the number circles represent the scatterers and the straight lines indicate propagation between the scatterers by $G_0(p)$. The second term in Eq. (3.13b) will be called the two-scatterer rattling ring and is illustrated in Fig. 1b. It will turn out that the two-scatterer rattling rings and the higher-order terms in Eq. (3.13b) will not be needed for the analysis presented here for the reduced density matrices.

If we now restrict ourselves to the two-scatterer ring diagrams the next approximation for $Z(\beta,n)$, denoted by $Z_2(\beta,n)$, is

$$Z_2(\beta,n) = \frac{1}{(2\pi\hbar)^3}\int d\mathbf{p}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{\beta s} \times\left[s + \frac{p^2}{2m} + n(2\pi\hbar)^3 t(\mathbf{p},\mathbf{p},s) + n^2\int d\mathbf{R}_1\int d\mathbf{R}_2 \times \hat{t}_1(\mathbf{p})G_0(p)\hat{t}_2(\mathbf{p})G_0(p)\hat{t}_1(\mathbf{p})\right]^{-1} \tag{3.18}$$

The use of Eqs. (3.8), (3.9), and (3.16) leads to an expression for the partition function of the form

$$Z_2(\beta,n) = \frac{1}{(2\eta\hbar)^3}\int d\mathbf{p}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{\beta s} \times\left[s + \frac{p^2}{2m} + n(2\pi\hbar)^3 t(\mathbf{p},\mathbf{p},s) + O\left(\frac{n^2}{\sqrt{s}}\right)\right]^{-1} \tag{3.19}$$

Equation (3.19) still cannot be used as it stands to compute the first density corrections to the value for $Z(\beta,n)$ given by Eq. (3.17a), for low temperatures. This can be seen from the fact that the low-temperature value of $Z_2(\beta,n)$ is determined by the small s value of the integrand in (3.19), and for small s, the term of $O(n^2/\sqrt{s})$ dominates the other contributions to $Z_2(\beta,n)$. Thus to obtain a useful expression for $Z(\beta,n)$ we will have to sum the most divergent (as $s\to 0$) terms in the integrand on the right-hand side of Eq. (3.14).

For the case of interest here this can be easily done because the l ($l = 2,3\ldots$) scatterer ring diagrams, illustrated in Fig. 1c, are the most divergent terms to each order in the density. The resulting resummation is illustrated diagrammatically in Fig. 1c and to carry it out we only need to

sum a geometric series. The resummed partition function, denoted by $Z_R(\beta,n)$, is given by

$$Z_R(\beta,n) = \frac{1}{(2\pi\hbar)^3}\int d\mathbf{p}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{\beta s} \times \left\{ s + \frac{p^2}{2m} + n(2\pi\hbar)^3 t(\mathbf{p},\mathbf{p},s) - n\int dR_1\, \hat{t}_1(\mathbf{p}) \times \left[\frac{1}{s + p^2/2m + n\int d\mathbf{R}_2\, \hat{t}_2(\mathbf{p})} - \frac{1}{s + p^2/2m} \right] \hat{t}_1(\mathbf{p}) \right\}^{-1} \tag{3.20}$$

Using Eqs. (3.8), (3.9), and (3.16) to lowest order in the scattering length, a, we obtain

$$Z_R(\beta,n) = \frac{1}{(2\pi\hbar)^3}\int dp\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{\beta s} \times \left\{ s + \frac{p^2}{2m} + n(2\pi\hbar)^3 t(\mathbf{p},\mathbf{p},s) - E_0 \frac{a(2ms)^{1/2}}{\hbar} + E_0 \frac{a(2m)^{1/2}}{\hbar}(s+E_0)^{1/2} \right\}^{-1} \tag{3.21}$$

with E_0 given by Eq. (3.17b). Inserting Eq. (3.16) into Eq. (3.21) we see that the terms proportional to $\sqrt{s}$ cancel and that $Z_R(\beta,n)$ can be written

$$Z_R(\beta,n) = \frac{e^{-n\lambda^2 a}}{(2\pi\hbar)^3}\int d\mathbf{p}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{\beta s} \times \left\{ s + \frac{p^2}{2m} + 2\sqrt{\pi}\, na^2\hbar\left(\frac{2\pi s}{m}\right)^{1/2} \right\}^{-1} + O\left[(na^2\lambda)^2, na^3\right] \tag{3.22}$$

where we have used the transformation $s' = s + E_0$. Giving s the weight $1/\beta$ we see that the corrections to Eq. (3.17b) are of $O(na^2\lambda)$ as expected.

By means of Eqs. (3.22) and (2.12a) and the fact that $\hat{g}(\hat{\mathbf{x}})$ is diagonal in the momentum representation, as is easily shown, the reduced density matrix $g(p)$ given by $\langle \mathbf{p}|\hat{g}(\hat{\mathbf{x}})|\mathbf{p}\rangle/\Omega = g(p)/(2\pi\hbar)^3$ can be computed to $O(na^2\lambda)$. To obtain $g(p)$ we use the same cluster expansion and approximations for the $\exp[-\beta\hat{H}(\hat{x},\mathbf{R}^N)]$ in Eq. (2.12a) as was used in calculating $Z(\beta,n)$. Proceeding along identical lines an expression for $g(p)$ correct to

$O(na^2\lambda)$ is

$$g(p) = \frac{\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{\beta s}\left\{ s + \frac{p^2}{2m} + 2\sqrt{\pi}\, na^2\hbar\left(\frac{2\pi s}{m}\right)^{1/2}\right\}^{-1}}{\frac{1}{(2\pi\hbar)^3}\int d\mathbf{p}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{\beta s}\left\{ s + \frac{p^2}{2m} + 2\sqrt{\pi}\, na^2\hbar\left(\frac{2\pi s}{m}\right)^{1/2}\right\}^{-1}} + O\left[(na^2\lambda)^2, na^3\right] \tag{3.23}$$

To $O(na^2\lambda)$ we can expand $\{\ \}^{-1}$ about the terms proportional to na^2. The term of $O(na^2\lambda)$ from $Z(\beta, n)$, the denominator in Eq. (3.23), vanishes identically, as can be shown by carrying out the indicated integrals, so that to $O(na^2\lambda)$

$$\begin{aligned} g(p) &= \lambda^3 e^{-\beta p^2/2m} - \frac{\lambda^3}{(2\pi i)} 2\pi na^2\hbar\left(\frac{2}{m}\right)^{1/2}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{\beta s} \\ &\quad \times \frac{\sqrt{s}}{\left[s + p^2/2m\right]^2} + O\left[(na^2\lambda)^2\right] \\ &= g_0(p) + g_1(p) + \cdots \end{aligned} \tag{3.24}$$

We postpone further evaluation of the term in Eq. (3.24) of $O(na^2\lambda)$ until Section 5 where it will be needed to calculate part of the first density correction of $O(na^2\lambda)$ to μ_B defined by Eq. (1.2).

The remaining reduced density matrices defined by Eqs. (2.12) can be calculated as a power series in $na^2\lambda$ by a similar manner. Because the procedure is straightforward we quote only the results here. The one-scatterer reduced density operator, $\hat{g}(\hat{\mathbf{x}}, \mathbf{R}_1)$, can be written [defining $\hat{h}(\hat{\mathbf{x}}, \mathbf{R}_1)$]:

$$\hat{g}(\hat{\mathbf{x}}, \mathbf{R}_1) = \hat{g}(\hat{\mathbf{p}}) + \hat{h}(\mathbf{x}, \mathbf{R}_1) \tag{3.25}$$

In our calculation of μ we will need the matrix elements of $\hat{h}(\hat{\mathbf{x}}, \mathbf{R}_1)$ only to lowest order in $na^2\lambda$. From Eqs. (2.12b), (3.1), (3.7a), and (3.16) we obtain

$$\langle \mathbf{p}|\hat{h}(\hat{\mathbf{x}}, \mathbf{R}_1)|\mathbf{p}_1\rangle = \exp\left[-(i\mathbf{R}_1/\hbar)\cdot(\mathbf{p} - \mathbf{p}_1)\right] h(\mathbf{p}, \mathbf{p}_1) \tag{3.26a}$$

where $[E(p) = p^2/2m]$

$$h(\mathbf{p}, \mathbf{p}_1) = \frac{\lambda^3}{(2\pi\hbar)^3}\,\frac{a\lambda^2}{\beta\left[E(p_1) - E(p)\right]}\left\{\exp\left[-\beta E(p_1)\right] - \exp\left[-\beta E(p)\right]\right\} \tag{3.26b}$$

with the corrections to Eq. (3.26b) being of relative order $na^2\lambda$. We also

need the two-scatterer reduced density operator, $\hat{g}(\hat{\mathbf{x}}, \mathbf{R}_1\mathbf{R}_2)$, which can be written [defining $\hat{h}(\hat{\mathbf{x}}, \mathbf{R}_1\mathbf{R}_2)$]:

$$\hat{g}(\hat{\mathbf{x}}, \mathbf{R}_1, \mathbf{R}_2) = \hat{g}(\hat{\mathbf{x}}) + \hat{h}(\hat{\mathbf{x}}, \mathbf{R}_1) + \hat{h}(\hat{\mathbf{x}}, \mathbf{R}_2) + \hat{h}(\hat{\mathbf{x}}, \mathbf{R}_1, \mathbf{R}_2) \tag{3.27}$$

with the matrix elements of $\hat{h}(\hat{\mathbf{x}}, \mathbf{R}_1, \mathbf{R}_2)$ being at least of $O(a^2)$. In our calculations of μ to $O(na^2\lambda)$ we will not need the explicit form of $\hat{h}(\hat{\mathbf{x}}, \mathbf{R}_1, \mathbf{R}_2)$.

Examining Eqs. (2.10) and (2.11) we see that the quantities involving the reduced density operators that appear in the calculation of μ are the commutator of these operators with $\hat{z}$, the operator for the z component of the moving particles position. Using $\langle \mathbf{p}|\hat{z}|\mathbf{p}_1\rangle = (\hbar/i)[(\partial/\partial p_{1z})\delta(\mathbf{p}_1 - \mathbf{p})]$ we obtain

$$\langle \mathbf{p}|\left[\hat{z}, \hat{g}(\hat{\mathbf{p}})\right]|\mathbf{p}_1\rangle = -\delta(\mathbf{p} - \mathbf{p}_1)\frac{\hbar}{i}\frac{\partial g(p)}{\partial p_z} \tag{3.28}$$

$$\langle \mathbf{p}|\left[\hat{z}, \hat{g}(\hat{\mathbf{p}})\right]^{-1}|\mathbf{p}_1\rangle = -\delta(\mathbf{p} - \mathbf{p}_1)\frac{i}{\hbar}\left[\frac{\partial g(p)}{\partial p_z}\right]^{-1} \tag{3.29}$$

$$\langle \mathbf{p}|\left[\hat{z}, \hat{g}(\hat{\mathbf{x}}, \mathbf{R}_1)\right]|\mathbf{p}_1\rangle = -\frac{\hbar}{i}\left\{\left[\frac{\partial}{\partial p_{1z}} + \frac{\partial}{\partial p_z}\right]h(\mathbf{p}, \mathbf{p}_1)\right\} \times \exp\left[-\frac{i\mathbf{R}_1}{\hbar}\cdot(\mathbf{p} - \mathbf{p}_1)\right] \tag{3.30}$$

In the next two sections of this paper these results as well as those in Section 2 will be used to derive Eq. (1.2).

4. THE ELECTRON MOBILITY IN THE LOW-DENSITY APPROXIMATION

In this section we use the results of Sections 2 and 3 to compute μ in the low-density or Boltzmann approximation. We first use the general expression for the electron mobility given by Eqs. (2.10) and (3.28) as

$$\mu = -\frac{e}{m(2\pi\hbar)^3}\lim_{\epsilon\to 0}\int d\mathbf{p}\,\Phi(\mathbf{p}, \epsilon)\frac{\partial g(p)}{\partial p_z} \tag{4.1}$$

since $\hat{\Phi}(\hat{\mathbf{p}}, \epsilon)$ and $[\hat{z}, \hat{g}(\hat{\mathbf{p}})]$ are diagonal in the momentum representation.

The low-density result for μ can be obtained from Eqs. (3.24), (4.1), and (2.13b) by keeping only the term of $O(n)$, $\hat{\mathfrak{B}}_1(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon)$, in Eq. (3.13b). Denoting this lowest-order approximation to $\hat{\Phi}(\hat{\mathbf{p}}, \epsilon)$ by $\hat{\Phi}_0(\hat{\mathbf{p}}, \epsilon)$ we obtain

from Eqs. (2.13b) and (2.14a) the operator kinetic equation

$$\Big\{\epsilon - n\int d\mathbf{R}_1\, \mathcal{G}_0^{-1}(\hat{\mathbf{p}})\big[\hat{z},\, \hat{g}(\hat{\mathbf{p}})\big]^{-1}\big[\hat{z},\, \hat{g}(\hat{\mathbf{x}}, \mathbf{R}_1)\big]$$

$$\times\, \hat{\mathcal{L}}_1(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon)\mathcal{G}_0^{-1}(\hat{\mathbf{p}})\Big\}\hat{\Phi}_0(\hat{\mathbf{p}}, \epsilon) = \hat{p}_z \tag{4.2}$$

From Eqs. (3.25) it follows that $[\hat{z},\, \hat{g}(\hat{\mathbf{p}})]^{-1}[\hat{z},\, \hat{g}(\hat{\mathbf{x}}, \mathbf{R}_1)] = 1 + [\hat{z},\, \hat{g}(\hat{\mathbf{p}})]^{-1}[\hat{z},\, \hat{h}(\hat{\mathbf{x}}, \mathbf{R}_1)]$. In the last part of this section we will show that the contribution proportional to $\hat{h}(\hat{\mathbf{x}}, \mathbf{R}_1)$ in the above expression gives a vanishing contribution to $\Phi(\hat{\mathbf{p}}, \epsilon)$ as $\epsilon \to 0$, and we consider first the term that does not depend on $\hat{h}(\hat{\mathbf{x}}, \mathbf{R}_1)$.

It will be useful here and in the next section to define a nonequilibrium binary collision operator[7] (actually a superoperator), $T(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon)$ in analogy with Eqs. (3.5a) and (3.7a),

$$T(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon) = \mathcal{G}_0^{-1}(\hat{\mathbf{p}})\hat{\mathcal{L}}_1(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon)\mathcal{G}_0^{-1}(\hat{\mathbf{p}})$$

$$= i\mathcal{V}(\hat{\mathbf{x}}, \mathbf{R}_1)\frac{1}{\big[\epsilon - i\mathcal{L}(\hat{\mathbf{x}}, \mathbf{R}_1)\big]}\big[\epsilon - i\mathcal{L}(\hat{\mathbf{p}})\big] \tag{4.3}$$

with

$$\mathcal{V}(\hat{\mathbf{x}}, \mathbf{R}_1) = \frac{1}{\hbar}\big[\,\hat{V}(\hat{\mathbf{r}} - \mathbf{R}_1),\ \big] \equiv \frac{1}{\hbar}\big[\,\hat{V}(\hat{\mathbf{x}}, \mathbf{R}_1),\ \big] \tag{4.4}$$

Using all this, Eq. (4.2) can be written

$$\Big[\epsilon - n\int d\mathbf{R}_1\, T(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon)\Big]\hat{\Phi}_0(\hat{\mathbf{p}}, \epsilon) = \hat{p}_z \tag{4.5}$$

A kinetic equation can be derived from Eq. (4.5) by taking matrix elements with respect to free particle states. This leads to

$$\epsilon\Phi_0(\mathbf{p}, \epsilon) - \frac{n(2\pi\hbar)^3}{\Omega}\int d\mathbf{R}_1 \langle\mathbf{p}|\big[\,T(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon)\hat{\Phi}_0(\hat{\mathbf{p}}, \epsilon)\big]|\mathbf{p}\rangle$$

$$= \epsilon\Phi_0(\mathbf{p}, \epsilon) - n\Lambda(\mathbf{p}, \epsilon)\Phi_0(\mathbf{p}, \epsilon) = p_z \tag{4.6}$$

where the second equality defines the collision operator $n\Lambda(\mathbf{p}, \epsilon)$. From Eq. (4.6) one can derive (as $\epsilon \to 0$) a quantum mechanical Boltzmann equation where all powers of the scattering length, a, appear through the quantum mechanical cross section $\sim|\langle\mathbf{p}|\hat{t}(\hat{\mathbf{x}}, \mathbf{R}_1, E(p))|\mathbf{p}_1\rangle|^2$, where $E(p) = p^2/2m$.[9] However, since the experimental region of interest is $a/\lambda \ll 1$ we will compute only the leading contribution in the scattering length expansion.

In Appendix B [cf. Eq. (B.3)] we show that the leading contribution to the matrix element in Eq. (4.6) is of $O(a^2)$ and that the s-wave $[\sim O(a^2)]$

quantum mechanical Boltzmann equation is

$$\epsilon\Phi_0(\mathbf{p},\epsilon) - n(2\pi\hbar)^3\left(\frac{a}{4\pi^2\hbar^2 m}\right)^2\int d\mathbf{p}_1 \frac{2\epsilon}{\left\{\epsilon^2 + (1/\hbar^2)\left[E(p) - E(p_1)\right]^2\right\}}$$

$$\times\left[\Phi(\mathbf{p}_1,\epsilon) - \Phi(\mathbf{p},\epsilon)\right] = p_z \tag{4.7}$$

This integral equation for $\Phi(\mathbf{p},\epsilon)$ can be solved by noting that the collision operator is an isotropic operator. A solution can then be obtained by setting $\Phi_0(\mathbf{p},\epsilon) = p_z\psi_0(p)$. As $\epsilon\to 0$ we obtain

$$\Phi_0(\mathbf{p},\epsilon) = \frac{m}{4\pi n a^2}\frac{p_z}{p} \tag{4.8}$$

From Eqs. (4.1), (4.8) and Eq. (3.24), and neglecting the term of $O(na^2\lambda)$ in Eq. (3.24), we can easily obtain the low-density value for μ, μ_B. The result is

$$\mu_B = -\frac{e}{m(2\pi\hbar)^3}\frac{m}{4\pi n a^2}\int d\mathbf{p}\frac{p_z}{p}\frac{\partial g_0(p)}{\partial p_z}$$

$$= \frac{\beta e}{3(2\pi\beta m)^{1/2} n\pi a^2} \tag{4.9}$$

which is the first term in Eq. (1.2). We remark that this result is equal to one-fourth the purely classical low-density mobility[10] since the s-wave hard sphere cross section is four times the classical hard-sphere cross section.

Next we verify that the term in Eq. (4.2) proportional to $\hat{h}(\hat{\mathbf{x}},\mathbf{R}_1)$ gives a vanishing contribution to $\Phi_0(\mathbf{p},\epsilon)$ as $\epsilon\to 0$. In terms of $T(\hat{\mathbf{x}},\mathbf{R}_1,\epsilon)$ the collision operator for this contribution is

$$\frac{n(2\pi\hbar)^3}{\Omega}\epsilon\int d\mathbf{R}_1\left\langle\mathbf{p}\middle|\left[\hat{z},\hat{\mathbf{g}}(\hat{\mathbf{p}})\right]^{-1}\left[\hat{z},\hat{h}(\hat{\mathbf{x}},\mathbf{R}_1)\right]\right.$$

$$\left.\times\left[\mathcal{G}_0(\hat{p})T(\hat{\mathbf{x}},\mathbf{R}_1,\epsilon)\hat{\Phi}_0(\hat{\mathbf{p}},\epsilon)\right]\middle|\mathbf{p}\right\rangle \tag{4.10}$$

where the factor ϵ originates from the term $\mathcal{G}_0^{-1}(\hat{p})$ to the left in Eq. (4.2). From the facts that $\hat{h}(\hat{\mathbf{x}},\mathbf{R}_1)$ is not diagonal in the momentum representation [cf. Eq. (3.26)] and that the matrix elements of $T(\hat{\mathbf{x}},\mathbf{R}_1,\epsilon)$ are well behaved as $\epsilon\to 0$ (cf. Appendix B) it follows that Eq. (4.10) vanishes as $\epsilon\to 0$.

In the next section we will compute corrections to Eq. (4.9) of relative order $na^2\lambda$. Since we have already neglected terms of $O(a/\lambda)$ the inequality $na^2\lambda > a/\lambda$ or $na\lambda^2 > 1$ must be satisfied for this to be a reasonable

procedure. As already mentioned this inequality is satisfied in typical experiments (see note 5 above).

5. THE FIRST DENSITY CORRECTION TO μ_B

In this section we will use the formalism developed in the previous sections to determine the first density correction to the Boltzmann result for the zero-field mobility. We will obtain the expression for (μ/μ_B) given earlier by Eq. (1.2), where terms of order unity and of order $(na^2\lambda)$ have been taken into account. We will consider higher-order terms in the following paper.

We begin by considering the kinetic equation for $\hat{\Phi}(\hat{\mathbf{p}}, \epsilon)$, Eq. (2.13b), and by using the fact that the first two terms in a density expansion of $\hat{\Phi}(\hat{\mathbf{p}}, \epsilon)$ are obtained by retaining only $\hat{\mathfrak{B}}_1$, and $\hat{\mathfrak{B}}_2$ on the right-hand side of Eq. (2.13b). If we denote the corresponding approximation to $\hat{\Phi}(\hat{\mathbf{p}}, \epsilon)$ by $\hat{\Phi}_1'(\hat{\mathbf{p}}, \epsilon)$, it follows that $\hat{\Phi}_1'(\hat{\mathbf{p}}, \epsilon)$ satisfies the equation

$$\left[\epsilon - n\int d\mathbf{R}_1\, \hat{\mathfrak{B}}_1(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon) - \frac{n^2}{2}\int d\mathbf{R}_1 \int d\mathbf{R}_2\, \hat{\mathfrak{B}}_2(\hat{\mathbf{x}}, \mathbf{R}_1, \mathbf{R}_2, \epsilon)\right]\hat{\Phi}_1'(\hat{\mathbf{p}}, \epsilon) = \hat{p}_z \tag{5.1}$$

where $\hat{\mathfrak{B}}_1$ and $\hat{\mathfrak{B}}_2$ are given by Eqs. (2.14a) and (2.14b), respectively.

The plan of this section is to solve Eq. (5.1) and to obtain μ to $O(na^2\lambda)$ by the following method: (1) We first determine the action of $\hat{\mathfrak{A}}_2(\hat{\mathbf{x}}_1, \mathbf{R}_1\mathbf{R}_2, \epsilon)$ appearing in Eq. (2.14b) for $\hat{\mathfrak{B}}_2$ by making a nonequilibrium binary collision expansion of it in terms of the T operator defined by Eq. (4.3). This (BCE) is formally analogous to that given in Section 3 for the two scatterer equilibrium cluster operator $\hat{u}_2(\hat{\mathbf{x}}, \mathbf{R}_1, \mathbf{R}_2, s)$. (2) We then order Eq. (5.1) in powers of the scattering length noting that terms of $O(n^2a^4\Phi)$ on the left-hand side of Eq. (5.1) are needed in the iterated solution of Eq. (5.1) around Φ_0 (cf. Section 4) to obtain corrections to μ_B of relative order $na^2\lambda$. (3) We then iterate the resulting equation around the solution $\hat{\Phi}_0(\hat{\mathbf{p}}_1, \epsilon)$ obtained in Section 4. (4) Three types of correction to μ_B of order $na^2\lambda$ are then identified and computed: (a) Corrections due to the replacement of $g_0(p)$ in Eq. (4.9) by the correct value to $O(na^2\lambda)$, $g_0(p) + g_1(p)$, where $g_1(p)$ is the contribution to the equilibrium single-particle distribution function from the equilibrium ring events defined in Section 3. (b) Contributions to μ of order $na^2\lambda$ due to dynamical events involving two scatterers and the moving particle described by those terms in the operator $\hat{\mathfrak{B}}_2$, that are independent of the equilibrium correlations $\hat{h}$, and (c) corrections which we will refer to as "static corrections" coming from the terms in $\hat{\mathfrak{B}}_2$ that are proportional to the equilibrium correlation $\hat{h}(\hat{\mathbf{x}}, \mathbf{R}^l)$.

We begin with the binary collision expansion of $\hat{\mathfrak{A}}_2(\hat{\mathbf{x}}, \mathbf{R}_1, \mathbf{R}_2, \epsilon)$, in

terms of T operators, given by

$$\begin{aligned}\hat{\mathfrak{A}}_2(\hat{\mathbf{x}}_1,\mathbf{R}_1,\mathbf{R}_2,\epsilon) = {} & (1+P_{12})\mathcal{G}_0(\hat{\mathbf{p}})T(\hat{\mathbf{x}}_1,\mathbf{R}_1,\epsilon)\mathcal{G}_0(\hat{\mathbf{p}})T(\hat{\mathbf{x}},\mathbf{R}_2,\epsilon)\mathcal{G}_0(\hat{\mathbf{p}}) \\ & + (1+P_{12})\mathcal{G}_0(\hat{\mathbf{p}})T(\hat{\mathbf{x}},\mathbf{R}_1,\epsilon)\mathcal{G}_0(\hat{\mathbf{p}})T(\hat{\mathbf{x}},\mathbf{R}_2,\epsilon)\mathcal{G}_0(\hat{\mathbf{p}}) \\ & \times T(\hat{\mathbf{x}},\mathbf{R}_1,\epsilon)\mathcal{G}_0(\hat{\mathbf{p}}) \\ & + (1+P_{12})\mathcal{G}_0(\hat{\mathbf{p}})T(\hat{\mathbf{x}},\mathbf{R}_1,\epsilon)\mathcal{G}_0(\hat{\mathbf{p}})T(\hat{\mathbf{x}},\mathbf{R}_2,\epsilon)\mathcal{G}_0(\hat{\mathbf{p}}) \\ & \times T(\hat{\mathbf{x}},\mathbf{R}_1,\epsilon)\mathcal{G}_0(\hat{\mathbf{p}})T(\hat{\mathbf{x}},\mathbf{R}_2,\epsilon)\mathcal{G}_0(\hat{\mathbf{p}}) + O(T^5) \qquad (5.2)\end{aligned}$$

where P_{ij} is a permutation operator that interchanges the scatterer labels i and j. We note that the term of $O(T^5)$ in Eq. (5.2) can be neglected since each T operator is at least of $O(a)$ and we need to keep only terms of $O(a^4)$ on the left-hand side of Eq. (5.1) to obtain the corrections to μ_B of $O(na^2\lambda)$.

To obtain a kinetic equation for $\hat{\Phi}'_1(\mathbf{p},\epsilon)$ we insert the binary collision expansion of $\hat{\mathfrak{A}}_2$, Eq. (5.2), into the expression for $\hat{\mathfrak{B}}_2$, Eq. (2.14b). In addition we use the cluster expansion of the $\hat{g}$'s given by Eqs. (3.25), (3.27), and the analysis of the operator $\hat{\mathfrak{B}}_1$ in Eqs. (4.3) and (4.6). Finally we take free particle matrix elements of the resulting form of Eq. (5.1) and we obtain

$$\left[\epsilon - n\Lambda(\mathbf{p},\epsilon)\right]\Phi'_1(\mathbf{p},\epsilon) - I_d(\mathbf{p},\epsilon;\Phi'_1) - I_{\mathrm{st}}(\mathbf{p},\epsilon;\Phi'_1) = p_z \qquad (5.3)$$

where the action of the quantum Lorentz–Boltzmann collision operator on Φ'_1 is given by [cf. Eq. (4.7)],

$$n\Lambda(\mathbf{p},\epsilon)\Phi'_1(\mathbf{p},\epsilon) = \frac{2na^2}{\pi m}\int d\mathbf{p}_1 \frac{2m\hbar\epsilon}{\left[(2m\hbar\epsilon)^2 + \left(p^2 - p_1^2\right)^2\right]}\left[\Phi'_1(\mathbf{p}_1,\epsilon) - \Phi'_1(\mathbf{p},\epsilon)\right] \qquad (5.4)$$

The contribution to this equation from dynamic events involving two scatterers of the moving particle is

$$\begin{aligned}I_d(\mathbf{p},\epsilon;\Phi'_1) = {} & \frac{n^2(2\pi\hbar)^3}{\Omega}\int d\mathbf{R}_1\int d\mathbf{R}_2 \\ & \times\langle\mathbf{p}|\{T(\hat{\mathbf{x}},\mathbf{R}_1,\epsilon)\mathcal{G}_0(\hat{\mathbf{p}})T(\hat{\mathbf{x}},\mathbf{R}_2,\epsilon)\mathcal{G}_0(\hat{\mathbf{p}})T(\hat{\mathbf{x}},\mathbf{R}_1,\epsilon) \\ & \times\left[1+\mathcal{G}_0(\hat{\mathbf{p}})T(\hat{\mathbf{x}},\mathbf{R}_2,\epsilon)\right]\hat{\Phi}'_1(\hat{\mathbf{p}},\epsilon)\}|\mathbf{p}\rangle \qquad (5.5)\end{aligned}$$

while the static contribution is given by

$$\begin{aligned}I_{\mathrm{st}}(\mathbf{p},\epsilon;\Phi'_1) = {} & -\frac{n^2(2\pi\hbar)^2}{\Omega}\int d\mathbf{R}_1\int d\mathbf{R}_2 \\ & \times\langle\mathbf{p}|\{T(\hat{\mathbf{x}},\mathbf{R}_1,\epsilon)\left[\hat{z},\hat{g}(\hat{\mathbf{p}})\right]^{-1}\left[\hat{z},\hat{h}(\hat{\mathbf{x}},\mathbf{R}_2)\right] \\ & \times\mathcal{G}_0(\hat{\mathbf{p}})T(\hat{\mathbf{x}},\mathbf{R}_2,\epsilon)\hat{\Phi}'_1(\hat{\mathbf{p}},\epsilon)\}|\mathbf{p}\rangle \qquad (5.6)\end{aligned}$$

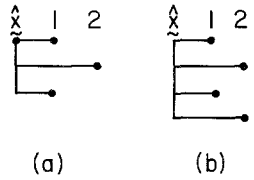

Fig. 2. (a) The nonequilibrium two-scatterer ring diagram. (b) The nonequilibrium two-scatterer rattling ring diagram.

Where in writing Eq. (5.3) we have neglected all terms with an $\hat{h}(\hat{\mathbf{x}}, \mathbf{R}^l)$ to the extreme left since these contributions are proportional to ϵ (cf. Section 4) and vanish as $\epsilon \to 0$, giving no contribution to the electron mobility.

It should be remarked that $I_d(\mathbf{p}, \epsilon; \Phi_1')$ is the sum of two types of dynamical operators acting on Φ_1' and they will be called the nonequilibrium two-scatterer ring contribution, involving a product of three T operators, and rattling ring contribution, involving four T operators, respectively, in analogy with the equilibrium interactions discussed in Section 3. Diagrammatically these contributions are given in Figs. 2a and 2b, respectively, where a horizontal line connecting $\hat{\mathbf{x}}$ and i ($i = 1, 2$) represents a $T(\hat{\mathbf{x}}, \mathbf{R}_i, \epsilon)$ operator and the vertical line segments under $\hat{\mathbf{x}}$ represent free particle propagation, $\mathcal{G}_0(\hat{\mathbf{p}})$.

Further, the terms I_d and I_{st} are of $O(a^4\Phi_1')$ as can be seen by noting that if a scatterer index i ($i = 1, 2$) appears in only one T operator and if there are no $\hat{h}$ operators with the same index then that T operator is at least of $O(a^2)$ since we integrate overall $\mathbf{R}_i$ (cf. Appendix B). Using this and Eq. (3.26b) it follows that I_d and I_{st} are at least of $O(a^4\Phi_1')$.

To calculate μ to $O(na^2\lambda)$ we need only solve Eq. (5.3) iteratively keeping the zeroth and first iterate around the solution of $[\epsilon - n\Lambda(\mathbf{p}, \epsilon)]$ $\Phi_0(\mathbf{p}, \epsilon) = p_z$. Writing

$$\Phi_1'(\mathbf{p}, \epsilon) = \Phi_0(\mathbf{p}, \epsilon) + \Phi_1(\mathbf{p}, \epsilon) + \Phi_2(\mathbf{p}, \epsilon) + \cdots \tag{5.7a}$$

and

$$\Phi_1(\mathbf{p}, \epsilon) = \Phi_{1d}(\mathbf{p}, \epsilon) + \Phi_{1\text{st}}(\mathbf{p}, \epsilon) \tag{5.7b}$$

the equations to be solved are

$$\big[\epsilon - n\Lambda(\mathbf{p}, \epsilon)\big]\Phi_0(\mathbf{p}, \epsilon) = p_z \tag{5.8}$$

$$\big[\epsilon - n\Lambda(\mathbf{p}, \epsilon)\big]\Phi_{1d}(\mathbf{p}, \epsilon) = I_d(\mathbf{p}, \epsilon; \Phi_0) \tag{5.9}$$

$$\big[\epsilon - n\Lambda(\mathbf{p}, \epsilon)\big]\Phi_{1\text{st}}(\mathbf{p}, \epsilon) = I_{\text{st}}(\mathbf{p}, \epsilon; \Phi_0) \tag{5.10}$$

where we have used the fact that these operators are linear operators to define Φ_{1d} and $\Phi_{1\text{st}}$ by Eqs. (5.9) and (5.10), respectively.

In terms of Φ_0 and Φ_1 the electron mobility is given by Eqs. (4.1) and

(3.24) to $O(na^2\lambda)$ as

$$\mu = \mu_B + \mu_{0st} + \mu_{1d} + \mu_{1st} \tag{5.11}$$

where μ_B is the Boltzmann contribution given by

$$\mu_B = -\frac{e}{m(2\pi\hbar)^3} \lim_{\epsilon\to 0} \int d\mathbf{p}\, \Phi_0(\mathbf{p},\epsilon) \frac{\partial g_0(p)}{\partial p_z} \tag{5.12}$$

μ_{0st} is obtained from Eq. (5.12) by replacing $g_0(p)$ by $g_1(p)$, the equilibrium ring correction to $g_0(p)$,

$$\mu_{0st} = -\frac{e}{m(2\pi\hbar)^3} \lim_{\epsilon\to 0} \int d\mathbf{p}\, \Phi_0(\mathbf{p},\epsilon) \frac{\partial g_1(p)}{\partial p_z} \tag{5.13}$$

μ_{1d} is given by

$$\mu_{1d} = -\frac{e}{m(2\pi\hbar)^3} \lim_{\epsilon\to 0} \int d\mathbf{p}\, \Phi_{1d}(\mathbf{p},\epsilon) \frac{\partial g_0(p)}{\partial p_z} \tag{5.14}$$

where we have used that to $O(na^2\lambda)$ we can replace $g(p)$ in this contribution by $g_0(p)$. Similarly μ_{1st} is given by

$$\mu_{1st} = -\frac{e}{m(2\pi\hbar)^3} \lim_{\epsilon\to 0} \int d\mathbf{p}\, \phi_{1st}(\mathbf{p},\epsilon) \frac{\partial g_0(p)}{\partial p_z} \tag{5.15}$$

These contributions will now be computed in order.

The Boltzmann contribution was calculated in Section 4 with the result given by Eq. (4.9). μ_{0st} can be computed from Eqs. (5.13) and the expansion of $g(p)$ given by Eq. (3.24). If we use the identity $\partial g_1(p)/\partial p_z = (\partial p/\partial p_z)(\partial g_1(p)/\partial p)$, and the expression for Φ_0, and then integrate Eq. (4.8), by parts, we obtain

$$\mu_{0st} = \frac{e}{(2\pi\hbar)^3} \frac{2}{3na^2} \int_0^\infty dp\, p g_1(p) \tag{5.16}$$

We next insert the expression for $g_1(p)$ from Eq. (3.24) into Eq. (5.16) interchanging the s and p integrals to obtain the final result for μ_{0st},

$$\mu_{0st} = -\mu_B 2na^2\lambda \tag{5.17}$$

Next we consider μ_{1d}. One can easily check that $I_d(\mathbf{p},\epsilon;\Phi_0)$ contains only isotropic operators, from which it follows that I_d, like Φ_0, is proportional to p_z. Using this, Eq. (5.9) can be solved by writing $\Phi_{1d}(\mathbf{p},\epsilon) = p_z\psi_{1d}(p)$. Then, as $\epsilon\to 0$, we obtain from Eqs. (5.4) and (5.9),

$$\Phi_{1d}(p,\epsilon) = \frac{m}{4\pi na^2 p} I_d(\mathbf{p},\epsilon;\Phi_0) \tag{5.18}$$

If this expression is inserted into Eq. (5.14), and we use Eq. (3.24) for $g_0(p)$,

we obtain

$$\mu_{1d} = \frac{\beta e}{m^2}\frac{\lambda^3}{(2\pi\hbar)^3}\frac{m}{4\pi n a^2}\int d\mathbf{p}\,\frac{p_z}{p}\,e^{-\beta p^2/2m}\lim_{\epsilon\to 0} I_d(\mathbf{p},\epsilon;\Phi_0) \tag{5.19}$$

To proceed with the evaluation of Eq. (5.19) we first note that $I_d(\mathbf{p},\epsilon;\Phi_0)$ (and hence μ_{1d}) can be written as the sum of two contributions:

$$I_d(\mathbf{p},\epsilon;\Phi_0) = I_d^v(\mathbf{p},\epsilon;\Phi_0) + I_d^r(\mathbf{p},\epsilon;\Phi_0) \tag{5.20}$$

$$\mu_{1d} = \mu_{1d}^v + \mu_{1d}^r \tag{5.21}$$

$I_d^v(\mathbf{p},\epsilon;\Phi_0)$ will be called the "virtual collision" part of I_d defined to be that part of I_d proportional to $\Phi_0(\mathbf{p},\epsilon)$, i.e., those terms in which the momentum $\mathbf{p}$ of the zeroth-order solution Φ_0 has not been changed by the action of the various collision operators. $I_d^r(\mathbf{p},\epsilon;\Phi_0)$ will be referred to as the "real collision" part of I_d and contains those contributions where Φ_0 is evaluated at a different momentum, $\mathbf{p}_1$. We first evaluate μ_{1d}^v.

In Appendix B we show that matrix elements of the binary collision operator $T(\hat{\mathbf{x}},\mathbf{R}_1,\epsilon)$ acting on an operator $\hat{A}$ can be expanded in powers of the scattering length a, and the first two terms are

$$\begin{aligned}
&\langle \mathbf{p}_1|\big[T(\hat{\mathbf{x}},\mathbf{R}_1,\epsilon)\hat{A}(\tilde{\mathbf{x}},\mathbf{R}^I)\big]|\mathbf{p}\rangle \\
&\quad = i\left(\frac{a}{4\pi^2\hbar^2 m}\right)\int d\mathbf{p}_2\left\{\exp\left[\frac{-i\mathbf{R}_1}{\hbar}\cdot(\mathbf{p}_1-\mathbf{p}_2)\right]\langle \mathbf{p}_2|\hat{A}(\hat{\mathbf{x}},\mathbf{R}^I)|\mathbf{p}\rangle\right. \\
&\qquad\qquad \left. -\exp\left[\frac{-i\mathbf{R}_1}{\hbar}\cdot(\mathbf{p}_2-\mathbf{p})\right]\langle \mathbf{p}_1|\hat{A}(\hat{\mathbf{x}},\mathbf{R}^I)|\mathbf{p}_2\rangle\right\} \\
&\quad + \left(\frac{a}{4\pi^2\hbar^2 m}\right)^2\int d\mathbf{p}_2\int d\mathbf{p}_3\exp\left[\frac{-i\mathbf{R}_1}{\hbar}\cdot(\mathbf{p}_1-\mathbf{p}_2+\mathbf{p}_3-\mathbf{p})\right] \\
&\quad \times\langle \mathbf{p}_2|\hat{A}(\hat{\mathbf{x}},\mathbf{R}^I)|\mathbf{p}_3\rangle \\
&\quad \times\left\{\frac{1}{\epsilon+(i/\hbar)\big[E(p)-E(p_2)\big]}+\frac{1}{\epsilon+(i/\hbar)\big[E(p_3)-E(p_1)\big]}\right\} \\
&\quad -\left(\frac{a^2}{4\pi^2\hbar^2 m}\right)\left(\frac{2m}{\hbar}\right)^{1/2}\int d\mathbf{p}_2 \\
&\quad \times\left\{\exp\left[\frac{-i\mathbf{R}_1}{\hbar}\cdot(\mathbf{p}_1-\mathbf{p}_2)\right]\left[\frac{E(p)-i\epsilon}{\hbar}\right]^{1/2}\langle \mathbf{p}_2|\hat{A}(\hat{\mathbf{x}},\mathbf{R}^I)|\mathbf{p}\rangle\right. \\
&\qquad \left. +\exp\left[\frac{-i\mathbf{R}_1}{\hbar}\cdot(\mathbf{p}_2-\mathbf{p})\right]\left[\frac{E(p_1)+i\epsilon}{\hbar}\right]^{1/2}\langle \mathbf{p}_1|\hat{A}(\hat{\mathbf{x}}_1,\mathbf{R}^I)|\mathbf{p}_2\rangle\right\} \\
&\quad + O(a^3)
\end{aligned} \tag{5.22}$$

With the aid of this and Eq. (4.8), a straightforward but lengthy algebraic manipulation yields

$$\mu_{1d}^{v} = -\frac{\beta e}{3\pi^{3/2}}\left(\frac{2\beta}{m}\right)^{3/2} m\hbar\,\mathrm{Re}\lim_{\epsilon\to 0}\int_0^\infty dp\, p^2 e^{-\beta p^2/2m}\int_0^\infty dp_1\, p_1^2$$
$$\times\left\{\frac{\left(p_1^2-2m\hbar i\epsilon\right)^{1/2}-\left(p^2-2m\hbar i\epsilon\right)^{1/2}}{\left[p_1^2-p^2\right]\left[p^2-p_1^2+2m\hbar i\epsilon\right]}\right.$$
$$\left.+\frac{\left(p^2-2m\hbar i\epsilon\right)^{1/2}+\left(p_1^2+2m\hbar i\epsilon\right)^{1/2}}{\left[p_1^2-p^2+2m\hbar i\epsilon\right]^2}\right\}+\tilde{\mu}_{1d}^{v} \tag{5.23}$$

where Re denotes the real part and $\tilde{\mu}_{1d}^{v}$ is given by

$$\tilde{\mu}_{1d}^{v} = \frac{\beta e}{6m^2}\left(\frac{2\pi\beta}{m}\right)^{3/2}\frac{\mathrm{Re}}{(2\pi)^4}\lim_{\epsilon\to 0}\int_0^\infty dp\, p^2 e^{-\beta p^2/2m}$$
$$\times\int_{-\infty}^{+\infty} dk\, k^2\int_0^\infty dp_1\, p_1^2\int_{-1}^{+1} d\sigma\int_{-1}^{+1} d\sigma_1$$
$$\times\left[\epsilon-\frac{i}{2m\hbar}\left(p_1^2-p^2\right)\right]^{-1}\left[\epsilon-\frac{ikp\sigma}{m}-\frac{i\hbar k^2}{2m}\right]^{-1}$$
$$\times\left[\epsilon-\frac{ikp_1\sigma_1}{m}-\frac{i\hbar k^2}{2m}-\frac{i}{2m\hbar}\left(p_1^2-p^2\right)\right]^{-1} \tag{5.24}$$

By carrying out the integrals in Eq. (5.23) one can easily show

$$\mu_{1d}^{v} = \tilde{\mu}_{1d}^{v} \tag{5.25}$$

Equation (5.24) can be evaluated by using the identity

$$\frac{1}{x-i\epsilon} = i\pi\delta(x)+\mathcal{P}\left(\frac{1}{x}\right) \tag{5.26}$$

where $\mathcal{P}$ denotes principal value. Using Eq. (5.26), scaling p_1 with p ($p_1 = xp$), and carrying out the p integral we obtain

$$\mu_{1d}^{v} = \mu_B\frac{na^2\lambda}{2}\int_0^\infty dx\, x^2\int_{-1}^{+1} d\sigma\int_{-1}^{+1} d\sigma_1$$
$$\times\left\{\frac{\theta\left[1-x^2\left(1-\sigma_1^2\right)\right]}{\left[1-x^2\left(1-\sigma_1^2\right)\right]^{1/2}}\left[2x\sigma\sigma_1+1-x^2\right]\right.$$
$$\times\mathcal{P}\left(\frac{1}{x^2-1}\right)\mathcal{P}\left[\frac{1}{4\sigma(\sigma-x\sigma_1)+x^2-1}\right]$$
$$\left.-2\mathcal{P}\left(\frac{1}{x^2-1}\right)|\sigma|\mathcal{P}\left[\frac{1}{4\sigma(\sigma-x\sigma_1)+x^2-1}\right]\right\} \tag{5.27}$$

where

$$\theta(x) = 1 \qquad \text{if} \quad x > 0$$
$$= 0 \qquad \text{otherwise} \tag{5.28}$$

and $|\sigma|$ denotes the absolute value of σ. The remaining integrals in Eq. (5.27) are all elementary and can be explicitly evaluated with the result

$$\mu_{1d}^{v} = -\mu_B n a^2 \lambda \frac{\pi^2}{2} \tag{5.29}$$

The evaluation of μ_{1d}^{r} is rather lengthy. We summarize the calculation in Appendix C, and merely quote the final result here:

$$\mu_{1d}^{r} = -\mu_B n a^2 \lambda \frac{\pi^2}{2} \tag{5.30}$$

Combining the results given by Eqs. (5.21), (5.29), and (5.30) we obtain

$$\mu_{1d} = -\mu_B n a^2 \lambda \pi^2 \tag{5.31}$$

We next determine $\mu_{1\text{st}}$. To do this we solve Eq. (5.10) and then use Eq. (5.15).

From Eqs. (5.22), (3.26a), and (5.6) we find that $I_{\text{st}}(\mathbf{p}, \epsilon; \Phi_0)$ is given to $O(a^4\Phi_0)$ by

$$\begin{aligned} I_{\text{st}}(\mathbf{p}, \epsilon; \Phi_0) = {} & -n^2(2\pi\hbar)^6 i \left(\frac{a}{4\pi^2\hbar^2 m} \right)^3 \int d\mathbf{p}_1 \int d\mathbf{p}_2 \frac{2\epsilon}{\epsilon^2 + \{[E(p) - E(p_2)]/\hbar\}^2} \\ & \times \left[\frac{\partial g_0(p_2)}{\partial p_{2z}} \right]^{-1} \left\{ \left[\frac{\partial}{\partial p_{2z}} + \frac{\partial}{\partial p_{1z}} \right] h(\mathbf{p}, \mathbf{p}_1) \right\} \\ & \times \frac{\Phi_0(\mathbf{p}_2, \epsilon) - \Phi_0(\mathbf{p}_1, \epsilon)}{\epsilon - (i/\hbar)[E(p_1) - E(p_2)]} \\ & + n^2(2\pi\hbar)^6 i \left(\frac{a}{4\pi^2\hbar^2 m} \right)^2 a \left(\frac{2m}{\hbar} \right)^{1/2} \left[\frac{\partial g_0(p)}{\partial p_z} \right]^{-1} \\ & \times \left\{ \left[\frac{E(p) - i\epsilon}{\hbar} \right]^{1/2} + \left[\frac{E(p) + i\epsilon}{\hbar} \right]^{1/2} \right\} \\ & \times \int d\mathbf{p}_1 \left\{ \left[\frac{\partial}{\partial p_z} + \frac{\partial}{\partial p_{1z}} \right] h(\mathbf{p}, \mathbf{p}_1) \right\} \frac{\Phi_0(\mathbf{p}, \epsilon) - \Phi_0(\mathbf{p}_1, \epsilon)}{\epsilon - (i/\hbar)[E(p_1) - E(p)]} \end{aligned} \tag{5.32}$$

In evaluating Eq. (5.32) and solving Eq. (5.10) ill-defined angular integrals like $\int d\hat{\mathbf{p}}/p_z$ are encountered. We interpret these integrals as principal value

integrals yielding $\int d\hat{\mathbf{p}}/p_z = 0$ with the remark that other reasonable interpretations[8] do not change the value obtained for the electron mobility. With this, the first term in Eq. (5.32) is found to vanish and with the second term on the right-hand side of Eq. (5.10) $\Phi_{1st}(\mathbf{p}, \epsilon)$ can be determined. Using the fact that $n\Lambda(\mathbf{p}, \epsilon)$ is an isotropic integral operator we obtain a value for μ_{1st}, from Eq. (5.15), given by

$$\mu_{1st} = -\frac{e}{m}(2\pi\hbar)^3 \frac{2n^2 ia}{\hbar}\left(\frac{m}{4\pi n a^2}\right)^2\left(\frac{a^2}{4\pi^2\hbar^2 m}\right)^2$$

$$\times \lim_{\epsilon\to 0}\int d\mathbf{p}\int d\mathbf{p}_1 \frac{1}{\epsilon - (i/\hbar)[E(p_1) - E(p)]}$$

$$\times\left[\frac{p_z}{p}\frac{\partial}{\partial p_z}h(\mathbf{p}, \mathbf{p}_1) - \frac{p_{1z}}{p_1}\frac{\partial}{\partial p_{1z}}h(\mathbf{p}, \mathbf{p}_1)\right] \tag{5.33}$$

with $h(\mathbf{p}, \mathbf{p}_1)$ given by Eq. (3.26b). The integrals in Eq. (5.33) can be evaluated without difficulty with the result

$$\mu_{1st} = \mu_B 2na^2\lambda \tag{5.34}$$

From Eqs. (5.11), (4.9), (5.17), (5.31), and (5.34) we obtain μ to $O(na^2\lambda)$:

$$\mu = \mu_B[1 - \pi^2 na^2\lambda] \tag{5.35}$$

which is the result quoted in the Introduction.

There may still be additional contributions to μ/μ_B of order $na^2\lambda$ which we have not included in Eq. (5.35). Such additional terms would come from terms in the binary collision expansion, Eq. (5.2), involving products of three or more T operators that would formally be of order a^5 or higher, while we have only taken into account terms of order a^4. The terms of order a^5, etc., would not contribute to the coefficient of $na^2\lambda$ in μ/μ_B if the coefficient of these terms were momentum integrals convergent at their upper limit. However, it is possible that the momentum integrals diverge if the upper limit is set equal to infinity, and the physical upper limit of $p_{max} = \hbar/a$ for s-wave scattering would need to be invoked to obtain finite results. From this we see that using the finite upper limits on the momentum integrals could produce additional factors of a^{-1}, etc. which taken with the powers of a from the T operators ultimately lead to corrections to μ/μ_B of order $na^2\lambda$. However, the contributions of these additional terms appear to be very small, typically on the order of a few percent of the total

[8] For example, if p_z is replaced by $p_z + i0^+$ then there is a delta function contribution $\sim\delta(p_z)$ that does not contribute to μ due to symmetry.

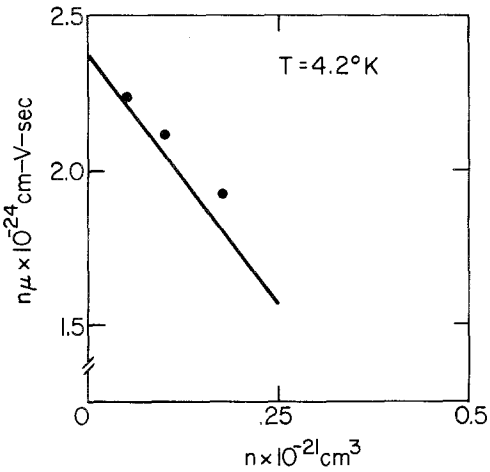

Fig. 3. The electron mobility times the density, $n\mu$, as a function of density for $T = 4.2°K$. The solid line here represents Eq. (1.2) and the dots represent the experimental data of Schwarz.[4]

contribution, and we will neglect them here. The relative smallness of these contributions follows from the fact that at large p the integrands of the momentum integrals are rapidly oscillating due to the appearance of spherical Bessel functions in the T operators, for large momenta (cf. Appendices A and B), leading to a small coefficient for these terms.

In Fig. 3 we have plotted Eq. (5.35) as well as the experimental results of Schwarz[4] for the electron mobility of helium at low temperatures. A discussion of this figure will be given in Section 6.

6. DISCUSSION

In this section we compare our results to those of previous workers and then conclude this paper by making some general remarks about the calculations presented here.

(1) In this paper we computed the first two terms in the density expansion of the electron mobility. Previous workers who have attempted similar calculations using binary collision expansion methods are Neustadter and Coopersmith,[11] and Braglia and Dallacasa,[12] and their calculations will be discussed in order.

The calculation that is closest in spirit to ours, and that leads to a number of interesting results, is that of Neustadter and Coopersmith. These authors attempt to compute the zero-field mobility of an electron in helium, which is modeled as a system of randomly placed stationary, hard-sphere

scatterers. They develop cumulant expansions for the canonical partition function for the electron in this system, and for zero-field mobility, and they make a scattering length expansion of the various cluster operators that appear in the cumulant expansion. By summing a large class of terms in these expansions, Coopersmith and Neustadter obtain an expression for the mobility that shows a pronounced decrease of several orders of magnitude over a small density range, and thus they seem to have provided a kinetic description for the "localization" of the electron in helium. Their results are also remarkably similar to a phenomenological description of localization due to Eggarter and Cohen.[13] Unfortunately, Coopersmith and Neustadter's results are not correct as they stand, nor are their results supported by the calculations carried out here. The central difficulty of their approach resides in the fact that they do not sum all the classes of terms in their cumulant expansion that must be taken into account for consistency. In fact the one class of terms they inconsistently neglected, when treated on the same footing with the others, precisely cancels the term that provided the effect of localization. To illustrate the cancellation, which occurs in both the partition function and the mobility, we outline Coopersmith's[11] calculation of the canonical partition function in Appendix D, and show that the ring diagrams discussed here in Section 3 cancel the important terms needed by the author for the description of localization. Therefore, in view of this cancellation, one must regard the kinetic description of the electron localization in helium as still an open problem which will require a more extensive analysis of the dynamical effects than has been given so far.

On a more modest scale, Braglia and Dallacasa[12] have carried out an evaluation of μ similar to that given here, where they look only for the first-order correction to the Boltzmann value. However, they took into account only the term that we have denoted by μ_{0st} in Section 5 of this paper. That is, they neglected both the dynamical and static correlations from $\hat{\mathfrak{B}}_3$ which must be consistently retained. Further, their calculation of μ_{0st} was also inconsistent since they used what corresponds to our Eq. (3.10) with $\hat{t}$ to $O(a^2)$ without at the same time performing the equilibrium ring diagram resummation. As shown in Section 3 all of these terms must be treated together.

(2) The most important difference between quantum mechanical kinetic theory and classical kinetic theory appears to be that the nonequilibrium binary collision operator in quantum mechanics has a contribution proportional to the scattering length a. Because of this, at low temperatures where a scattering length expansion can be used this purely quantum mechanical part of the T operator is the dominant part. In fact, it is responsible for the term of $O(na^2\lambda)$ in μ.

(3) In Section 5 we solved Eq. (5.3) iteratively and showed that the total contribution to the term of $O(na^2\lambda)$ in Eq. (1.2) is due to the dynamical collision sequences illustrated in Fig. 2, i.e., all effects due to static or equilibrium correlations are canceled. Here we note that if $I_{st}(\mathbf{p}, \epsilon; \Phi_1')$ is neglected in Eq. (5.3) from the beginning then the resulting integral equation can be solved exactly. To show this one needs

$$n\Lambda(\mathbf{p}, 0)\psi(p)p_z = \psi(p)pn\Lambda(\mathbf{p}, 0)\hat{p}_z \tag{6.1}$$

$$I_d(\mathbf{p}, 0; \psi(p)p_z) = \psi(p)pI_d(\mathbf{p}, 0; \hat{p}_z) \tag{6.2}$$

where $\psi(p)$ is an arbitrary function of $|\mathbf{p}|$ and $\hat{p}_z = p_z/|\mathbf{p}|$. Equation (6.1) follows directly from Eq. (5.4) and Eq.(6.2) can be verified by using Eqs. (5.5) and (5.22). Further, we believe that properties similar to Eqs. (6.1) and (6.2) are valid to all orders in the density and in the scattering length since an isotropic kinetic equation should (as $\epsilon \to 0$) transform an incoming state with momentum $\mathbf{p}$ into an asymptotic outgoing state with momentum $\mathbf{p}'$ with the restriction $|\mathbf{p}| = |\mathbf{p}'|$. We note that collision operators which contain static correlations, e.g., $I_{st}(\mathbf{p}, \epsilon; \Phi_1')$, are not isotropic operators due to the commutators $[\hat{z}, \ ]$ that they contain, cf. Eq. (5.6). Consequently, they do not satisfy a property similar to Eq. (6.2) and do not conserve $|\mathbf{p}|$.

Using Eqs. (6.1), (6.2), (5.4), (5.5), and (5.22) in Eq. (5.3), neglecting $I_{st}(\mathbf{p}, \epsilon; \Phi_1')$, we obtain

$$\Phi_1'(\mathbf{p}, 0) = \frac{m}{4\pi na^2} \frac{p_z}{\left[p + 4\pi^2 na^2\hbar\right]} \tag{6.3}$$

Inserting Eq. (6.3) into Eq. (4.1), consistently replacing $g(p)$ by $g_0(p)$, and expanding $[p + 4\pi^2 na^2\hbar]^{-1}$ in $\Phi_1'(\mathbf{p}, 0)$ about p^{-1} yields Eq. (1.2) plus corrections of $O[(na^2\lambda)^2]$.

(4) In Fig. 3 we compare our results for μ with the experimental data of Schwarz,[4] and find reasonable agreement between theory and experiment for densities not too high. We note that a horizontal line in this figure would represent a mobility, μ_B, given by the Lorentz–Boltzmann equation.

APPENDIX A. DERIVATION OF THE t-MATRIX

In this appendix we calculate $t_1(\mathbf{p}, \mathbf{p}_1, s; \mathbf{R}_1)$ defined by Eq. (3.5) for the hard-sphere interactions given by Eqs. (2.4) and (2.5). Since similar calculations have been given elsewhere[14] we will only sketch the calculation here. Further, since in Appendix B we will need $t_1(\mathbf{p}, \mathbf{p}_1, -\hbar\omega; \mathbf{R}_1)$ we calculate here $t_1(\mathbf{p}, \mathbf{p}_1, -(z \pm i0); \mathbf{R}_1) \equiv t_1^{(\pm)}(\mathbf{p}, \mathbf{p}_1, -z; \mathbf{R}_1)$ in the complex z-plane cut along the positive real axis.

To determine $t_1^{(\pm)}(\mathbf{p}, \mathbf{p}_1, -z; \mathbf{R}_1)$ we first define a ket vector, $|\psi\rangle^{(\pm)}$, that satisfies the Schrödinger equation

$$\left[z \pm i0 - \hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}_1)\right]|\psi\rangle^{(\pm)} = |\mathbf{p}_1\rangle \tag{A.1}$$

Using Eq. (3.5), we can express the t matrix in terms of $\langle \mathbf{p} \,|\, \psi\rangle^{(\pm)}$ by

$$\langle \mathbf{p} \,|\, \psi\rangle^{(\pm)} = \frac{\delta(\mathbf{p} - \mathbf{p}_1)}{\left[z \pm i0 - E(p)\right]} + \frac{1}{\left[z \pm i0 - E(p)\right]} \times \langle \mathbf{p}|\hat{t}^{(\pm)}(\hat{\mathbf{x}}, \mathbf{R}_1, -z)|\mathbf{p}_1\rangle \frac{1}{\left[z \pm i0 - E(p_1)\right]} \tag{A.2}$$

The plan of this appendix is to solve Eq. (A.1) in the coordinate representation, i.e., solve the equation

$$\left[z \pm i0 + \frac{\hbar^2}{2m}\nabla_r^2 - V_d(\mathbf{r} - \mathbf{R}_1)\right]\langle \mathbf{r} \,|\, \psi\rangle^{(\pm)} = \frac{\exp\left[i\mathbf{p}_1 \cdot \mathbf{r}/\hbar\right]}{(2\pi\hbar)^{3/2}} \tag{A.3}$$

and then use

$$\langle \mathbf{p} \,|\, \psi\rangle^{(\pm)} = \frac{1}{(2\pi\hbar)^{3/2}} \int d\mathbf{r} \exp\left[-i\mathbf{p} \cdot \mathbf{r}/\hbar\right]\langle \mathbf{r} \,|\, \psi\rangle^{(\pm)} \tag{A.4}$$

in Eq. (A.2) to determine $\langle \mathbf{p}|\hat{t}^{(\pm)}(\hat{\mathbf{x}}, \mathbf{R}_1, -z)|\mathbf{p}_1\rangle$.

Equation (A.3) is the Schrödinger equation and for the hard-sphere interaction given by Eq. (2.5) its solution can be easily obtained in terms of partial waves.[5] The result is[9]

$$\langle \mathbf{r} \,|\, \psi\rangle^{(\pm)} = \frac{W(|\mathbf{r} - \mathbf{R}_1|)}{\left[z \pm i0 - E(p_1)\right]} \frac{\exp\left[i\mathbf{p}_1 \cdot \mathbf{R}_1/\hbar\right]}{(2\pi\hbar)^{3/2}} \times \left\{ \exp\left[i\mathbf{p}_1 \cdot (\mathbf{r} - \mathbf{R}_1)/\hbar\right] - \sum_{l=0}^{\infty} (2l+1) i^l \times P_l(\cos\theta') \frac{j_l(p_1 a/\hbar)}{h_l^{(\pm)}(\zeta a)} h_l^{(\pm)}(\zeta|\mathbf{r} - \mathbf{R}_1|) \right\} \tag{A.5}$$

Here $W(|\mathbf{r} - \mathbf{R}_1|)$ is a characteristic function that vanishes for $|\mathbf{r} - \mathbf{R}| \leqslant a$, and is unity otherwise, P_l is the Legendre function, θ' is the angle between

[9] It should be remembered, that in solving Eq. (A.3) for hard-core potentials there is a problem. That is, for these potentials we must have $\langle \mathbf{r} \,|\, \psi\rangle = 0$ for $|\mathbf{r} - \mathbf{R}_1| \leqslant a$ which is not consistent with the right-hand side of Eq. (A.3). This difficulty can be overcome by incorporating $W(|\mathbf{r} - \mathbf{R}_1|)$ functions into the definition of the t matrices as in the classical case for the T operators.[15] We do not do this here since the effect of the inconsistency in Eq. (A.3) leads to excluded volume corrections of $O(na^3)$ which can be neglected here.

$\mathbf{p}_1$ and $\mathbf{r} - \mathbf{R}_1$, j_l is the spherical Bessel function, $h_l^{(\pm)}$ is the spherical Hankel function of the first (+) and second (−) kind, and $\zeta = [2m(z \pm i0)/\hbar^2]^{1/2}$. Using Eqs. (A.2), (A.4), and (A.5) we obtain

$$\begin{aligned}\langle \mathbf{p}|\hat{t}^{(\pm)}(\hat{\mathbf{x}}, \mathbf{R}_1, -z)|\mathbf{p}_1\rangle &\equiv \exp\left[-i\mathbf{R}_1 \cdot (\mathbf{p} - \mathbf{p}_1)/\hbar\right] t^{(\pm)}(\mathbf{p}, \mathbf{p}_1 - z) \\ &= -\frac{\left[z \pm i0 - E(p)\right]}{(2\pi\hbar)^3} \exp\left[-i\mathbf{R}_1 \cdot (\mathbf{p} - \mathbf{p}_1)/\hbar\right] \\ &\quad \times \left\{ \int_{|\mathbf{r}| \leqslant a} d\mathbf{r} \exp\left[-i\mathbf{r} \cdot (\mathbf{p} - \mathbf{p}_1)\hbar\right] \right. \\ &\quad + \sum_{l=0}^{\infty} (2l+1) i^l \frac{j_l(p_1 a/\hbar)}{h_l^{(\pm)}(\zeta a)} \\ &\quad \left. \times \int d\mathbf{r} \exp\left[-i\mathbf{p} \cdot \mathbf{r}/\hbar\right] W(r) P_l(\cos\theta') h_l^{(\pm)}(\zeta r) \right\}\end{aligned} \tag{A.6}$$

Examining Eq. (A.6) we see that Eq. (3.9) has been verified.

$t^{(\pm)}(\mathbf{p}, \mathbf{p}_1, -z)$ can be computed by carrying out the $\mathbf{r}$ integrals with the result

$$\begin{aligned}t^{(\pm)}(p, p_1, -z) &= -\left[z \pm i0 - E(p)\right] \frac{a^2}{2\pi^2\hbar^2} \frac{j_1(a|\mathbf{p} - \mathbf{p}_1|/\hbar)}{|\mathbf{p} - \mathbf{p}_1|} \\ &\quad + \frac{a^2}{4\pi^2\hbar m} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \\ &\quad \times \left[\zeta j_l\left(\frac{p_1 a}{\hbar}\right) j_l\left(\frac{p_a}{\hbar}\right) \frac{h_{l+1}^{(\pm)}(\zeta a)}{h_l^{(\pm)}(\zeta a)} - \frac{p}{\hbar} j_{l+1}\left(\frac{pa}{\hbar}\right) j_l\left(\frac{p_1 a}{\hbar}\right) \right]\end{aligned} \tag{A.7}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{p}_1$. To $O(a^2)t^{(\pm)}(\mathbf{p}, \mathbf{p}_1, -z)$ is given by

$$t^{(\pm)}(\mathbf{p}, \mathbf{p}_1, -z) = \frac{a}{4\pi^2\hbar m} \left\{ 1 \mp i \frac{a}{\hbar} \left[2m(z \pm i0)\right]^{1/2} \right\} + O(a^3) \tag{A.8}$$

APPENDIX B. DERIVATION OF THE T-OPERATOR

In this appendix we derive the representations of the nonequilibrium binary collision operator given by Eqs. (5.22) and (4.7).

From Eqs. (4.3) and (4.4) the left-hand side of Eq. (5.22) can be

written

$$\langle \mathbf{p}_1 | \big[T(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon) A(\hat{\mathbf{x}}, \mathbf{R}^l) \big] | \mathbf{p} \rangle$$

$$= -i \int_0^\infty dt \frac{e^{-\epsilon t}}{(2\pi i)^2} \int_c d\omega \int_{c'} d\omega' \, e^{-i(\omega+\omega')t}$$

$$\times \Bigg\{ \langle \mathbf{p}_1 | \frac{\hat{V}}{\hbar} (\mathbf{x}, \mathbf{R}_1) \frac{1}{\big[(\hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}_1)/\hbar) + \omega \big]}$$

$$\times \hat{B}(\hat{\mathbf{x}}, \mathbf{R}^l) \frac{1}{\big[(\hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}_1)/\hbar) - \omega' \big]} | \mathbf{p} \rangle$$

$$- \langle \mathbf{p}_1 | \frac{1}{\big[(\hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}_1)/\hbar) + \omega \big]} \hat{B}(\hat{x}, R^l)$$

$$\times \frac{1}{\big[(\hat{H}_d(\hat{\mathbf{x}}, \mathbf{R}_1)/\hbar) - \omega' \big]} \frac{\hat{V}}{\hbar} (\hat{\mathbf{x}}, \mathbf{R}_1) | \mathbf{p} \rangle \Bigg\} \tag{B.1}$$

where $\hat{B}(\hat{\mathbf{x}}, \mathbf{R}^l) = [\epsilon - i\mathfrak{L}(\hat{\mathbf{p}})]\hat{A}(\hat{\mathbf{x}}, \mathbf{R}^l)$ and $c(c')$ is a contour in the complex $\omega(\omega')$ plane that passes above the real axis and is closed in the lower half plane. Equation (B.1) can be easily expressed in terms of the t matrix defined by Eq. (3.5) as

$$\langle \mathbf{p}_1 | \big[T(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon) \hat{A}(\hat{\mathbf{x}}, \mathbf{R}^l) \big] | \mathbf{p} \rangle$$

$$= -i\hbar \int_0^\infty dt \, e^{-\epsilon t} \int_c d\omega \int_{c'} d\omega' \, e^{-i(\omega+\omega')t}$$

$$\times \int d\mathbf{p}_2 \times \Bigg\{ \frac{\langle \mathbf{p}_1 | \hat{t}(\hat{\mathbf{x}}, \mathbf{R}_1, \hbar\omega) | \mathbf{p}_2 \rangle \langle \mathbf{p}_2 | \hat{B}(\hat{\mathbf{x}}, \mathbf{R}^l) | \mathbf{p} \rangle}{\big[E(p_2) + \hbar\omega \big] \big[E(p) - \hbar\omega' \big]}$$

$$- \frac{\langle \mathbf{p}_1 | \hat{B}(\hat{\mathbf{x}}, \mathbf{R}^l) | \mathbf{p}_2 \rangle \langle \mathbf{p}_2 | \hat{t}(\hat{\mathbf{x}}, \mathbf{R}_1, -\hbar\omega') | \mathbf{p} \rangle}{\big[E(p_1) + \hbar\omega \big] \big[E(p_2) - \hbar\omega' \big]}$$

$$- \int d\mathbf{p}_3 \frac{\langle \mathbf{p}_1 | \hat{t}(\hat{\mathbf{x}}, \mathbf{R}_1, \hbar\omega) | \mathbf{p}_2 \rangle \langle \mathbf{p}_2 | \hat{B}(\hat{\mathbf{x}}, \mathbf{R}^l) | \mathbf{p}_3 \rangle \langle \mathbf{p}_3 | \hat{t}(\hat{\mathbf{x}}, \mathbf{R}_1, -\hbar\omega') | \mathbf{p} \rangle}{\big[E(p_2) + \hbar\omega \big] \big[E(p_3) - \hbar\omega' \big]}$$

$$\times \Bigg[\frac{1}{E(p) - \hbar\omega'} - \frac{1}{E(p_1) + \hbar\omega} \Bigg] \Bigg\} \tag{B.2}$$

Using Eqs. (A.6) and (A.8) in Eq. (B.2) and carrying out the ω, ω' and t integrals Eq. (5.22) can be obtained without difficulty.

We note that in deriving the quantum mechanical Boltzmann equation, discussed in Section 4, Eq. (B.2) appears with $\mathbf{p}_1 = \mathbf{p}$ and with $\hat{A} = \hat{A}(\hat{\mathbf{p}})$ diagonal in the momentum representation. For this case $\langle \mathbf{p}_2|\hat{B}(\hat{\mathbf{x}}, \mathbf{R}')|\mathbf{p}\rangle = \epsilon A(\mathbf{p})\delta(\mathbf{p} - \mathbf{p}_2)$ and Eq. (B.2) can be written

$$\begin{aligned}
&\langle \mathbf{p}|\big[\, T(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon)\hat{A}(\hat{\mathbf{p}})\,\big]|\mathbf{p}\rangle \\
&\quad = -i\hbar\epsilon \int_0^\infty dt\, e^{-\epsilon t} \int_c d\omega \int_{c'} d\omega'\, e^{-i(\omega+\omega')t} \int d\mathbf{p}_1 \\
&\qquad \times \frac{t(\mathbf{p}, \mathbf{p}_1, \hbar\omega)\, t(\mathbf{p}_1, \mathbf{p}, -\hbar\omega')}{\big[E(p_1) + \hbar\omega\big]\big[E(p_1) - \hbar\omega'\big]} \left[\frac{1}{E(p) - \hbar\omega'} - \frac{1}{E(p) + \hbar\omega} \right] \\
&\qquad \times \big[A(\mathbf{p}') - A(\mathbf{p})\big] \qquad \text{(B.3)}
\end{aligned}$$

where Eq. (A.6) has been used as well as the generalized optical theorem:

$$\begin{aligned}
&\langle \mathbf{p}|\big[\, t(\mathbf{x}, \mathbf{R}_1, \hbar\omega) - t(\mathbf{x}, \mathbf{R}_1, -\hbar\omega')\,\big]|\mathbf{p}\rangle \\
&\quad = \hbar(\omega + \omega') \int d\mathbf{p}_1 \frac{\langle \mathbf{p}|\hat{t}(\hat{\mathbf{x}}, \mathbf{R}_1, \hbar\omega)|\mathbf{p}_1\rangle \langle \mathbf{p}_1|\hat{t}(\hat{\mathbf{x}}, \mathbf{R}_1, -\hbar\omega')|\mathbf{p}\rangle}{\big[E(p_1) + \hbar\omega\big]\big[E(p_1) - \hbar\omega'\big]} \qquad \text{(B.4)}
\end{aligned}$$

Equation (B.4) can be easily derived from the definition of t given by Eq. (3.5a). From Eqs. (A.8), (B.3) and Eq. (4.6), Eq. (4.7) can be obtained

APPENDIX C. DERIVATION OF EQ. (5.30)

In this appendix we sketch the derivation of Eq. (5.30) for μ_{1d}^r. From Eqs. (5.22) and (4.8) straightforward but lengthy algebraic manipulation yields

$$\begin{aligned}
\mu_{1d}^r = &-\frac{\beta e}{6m^2}\left(\frac{2\pi\beta}{m}\right)^{3/2} \frac{\mathrm{Re}}{(2\pi)^4} \lim_{\epsilon\to 0} \int_0^\infty dp\, p^2 e^{-\beta p^2/2m} \\
&\times \int_0^\infty dp_1\, p_1^2 \int_{-\infty}^{+\infty} dk\, d^2 \int_{-1}^{+1} d\sigma\, \sigma \int_{-1}^{+1} d\sigma_1\, \sigma_1 \\
&\times \left\{ \left[\epsilon - \frac{ikp\sigma}{m} - \frac{i\hbar k^2}{2m} \right]^{-1} \left[\epsilon - \frac{i}{2m\hbar}\left(p_1^2 - p^2\right) \right]^{-1} \right. \\
&\quad \times \left[\epsilon - \frac{i\hbar k^2}{2m} - \frac{ikp_1\sigma_1}{m} - \frac{i}{2m\hbar}\left(p_1^2 - p^2\right) \right]^{-1} \\
&\quad + \left[\epsilon - \frac{i}{2m\hbar}\left(p^2 - p_1^2\right) \right]^{-1} \left[\epsilon - \frac{i}{2m\hbar}\left(p^2 - p_1^2\right) - \frac{ikp\sigma}{m} - \frac{i\hbar k^2}{2m} \right]^{-1}
\end{aligned}$$

$$\times\left[\epsilon - \frac{i\hbar k^2}{2m} - \frac{ikp_1\sigma_1}{m}\right]^{-1}$$

$$-\left[\epsilon - \frac{i}{2m\hbar}(p^2 - p_1^2) + \frac{i\hbar k^2}{2m} - \frac{ikp_1\sigma_1}{m}\right]^{-1}$$

$$\times\left[\epsilon - \frac{i}{2m\hbar}(p^2 - p_1^2) - \frac{ikp\sigma}{m} - \frac{ikp_1\sigma_1}{m}\right]^{-1}\left[\epsilon + \frac{i\hbar k^2}{2m} - \frac{ikp_1\sigma_1}{m}\right]^{-1}$$

$$+\left[\epsilon - \frac{ikp\sigma}{m} - \frac{i\hbar k^2}{2m}\right]^{-1}\left[\epsilon - \frac{ikp_1\sigma_2}{m} - \frac{i\hbar k^2}{2m}\right]^{-1}$$

$$\times\left[\epsilon - \frac{ikp_1\sigma_1}{m} - \frac{i\hbar k^2}{2m} - \frac{i}{2m\hbar}(p_1^2 - p^2)\right]^{-1}$$

$$+\left[\epsilon - \frac{ikp\sigma}{m} - \frac{i\hbar k^2}{2m}\right]^{-1}\left[\epsilon - \frac{ikp_1\sigma_1}{m} - \frac{i\hbar k^2}{2m}\right]^{-1}$$

$$\times\left[\epsilon - \frac{ikp\sigma}{m} - \frac{i\hbar k^2}{2m} + \frac{i}{2m\hbar}(p_1^2 - p^2)\right]^{-1}$$

$$-\left[\epsilon - \frac{ikp\sigma}{m} - \frac{i\hbar k^2}{2m}\right]^{-1}\left[\epsilon - \frac{ikp_1\sigma_1}{m} + \frac{i\hbar k^2}{2m}\right]^{-1}$$

$$\times\left[\epsilon - \frac{i}{2m\hbar}(p_1^2 - p^2)\right]^{-1} - \left[\epsilon - \frac{ikp\sigma}{m} - \frac{i\hbar k^2}{2m}\right]^{-1}$$

$$\times\left[\epsilon - \frac{ikp_1\sigma_1}{m} + \frac{i\hbar k^2}{2m}\right]^{-1}$$

$$\times\left[\epsilon - \frac{ikp\sigma}{m} - \frac{ikp_1\sigma_1}{m} + \frac{i}{2m\hbar}(p_1^2 - p^2)\right]^{-1}\Bigg\} \tag{C.1}$$

where in giving this expression we have used identities like

$$\int d\hat{\mathbf{p}}\, p_z\left[\epsilon - \frac{i\mathbf{k}\cdot\mathbf{p}}{m} - \frac{i\hbar k^2}{2m}\right]^{-1} = \frac{2\pi k_z p}{k}\int_{-1}^{+1} d\sigma\,\sigma\left[\epsilon - \frac{ikp\sigma}{m} - \frac{i\hbar k^2}{2m}\right]^{-1} \tag{C.2}$$

where $d\hat{\mathbf{p}}$ is the angular part of $d\mathbf{p}$. By repeated use of Eq. (5.26), scaling p_1

with p ($p_1 = xp$), and carrying out the p integral we obtain

$$\mu_{1d}^{r} = -\mu_B \frac{na^2\lambda}{2} \int_0^\infty dx\, x^2 \int_{-1}^{+1} d\sigma\, \sigma \int_{-1}^{+1} d\sigma_1\, \sigma_1$$

$$\times \Bigg\{ 3\pi^2 \delta(x^2-1)\delta(\sigma-\sigma_1) + 2\mathcal{P}$$

$$\times \left(\frac{1}{x^2-1}\right)|\sigma|^{\mathcal{P}} \left[\frac{1}{4\sigma(\sigma - x\sigma_1) + x^2 - 1}\right] + \frac{\theta\left[1 - x^2(1-\sigma_1^2)\right]}{\left[1 - x^2(1-\sigma_1^2)\right]^{1/2}}$$

$$\times \left[2x\sigma\sigma_1 + 1 - x^2\right]^{\mathcal{P}} \left(\frac{1}{x^2-1}\right)^{\mathcal{P}} \left[\frac{1}{4\sigma(\sigma - x\sigma_1) + x^2 - 1}\right] \Bigg\} \quad \text{(C.3)}$$

The remaining integrals in Eq. (C.3) are elementary and can be straightforwardly evaluated. The last two terms are found to precisely cancel and the first term gives Eq. (5.30). We believe the above cancellation is a general feature (cf. Section 6.3) of the calculations and is related to Bég's theorem used in quantum mechanical scattering theory (see, for example, Ref. 16).

APPENDIX D

In this appendix we outline the calculation of the canonical partition function for an electron in a random system of fixed, hard-sphere scatterers, as given by Coopersmith.[11] We then show that the incorporation of the additional terms that must be retained for consistency leads to a cancellation of the terms needed by Coopersmith for his description of electron localization.

We begin by writing the canonical partition function, Z, as

$$Z = \lim_{\substack{N\to\infty \\ \Omega\to\infty \\ N/\Omega = n}} \mathrm{Tr} \int \frac{d\mathbf{R}^N}{\Omega^{N+1}} \exp\left[-\beta \hat{H}(\hat{\mathbf{x}}, \mathbf{R}^N)\right] = \int \frac{d\mathbf{p}\, e^{-\beta p^2/2m}}{(2\pi\hbar)^3} Z_p \quad \text{(D.1)}$$

where Z_p can be expressed in terms of equilibrium cluster operators $\hat{u}(\hat{\mathbf{x}}, \mathbf{R}^l, \beta)$, Eq. (3.2), by

$$Z_p = 1 + C_p^{-1} \sum_{l=1}^{\infty} \frac{n^l}{l!} \int d\mathbf{R}^l \langle \mathbf{p}|\hat{u}(\hat{\mathbf{x}}, \mathbf{R}^l, \beta)|\mathbf{p}\rangle \quad \text{(D.2)}$$

where $C_p = \Omega e^{-\beta p^2/2m}(2\pi\hbar)^{-3}$. We next carry out a cumulant expansion of

Z_p by writing

$$Z_p = \exp\left[\sum_{l=1}^{\infty} n^l B_l(\mathbf{p})\right] \tag{D.3}$$

By expanding the right-hand side of Eq. (D.3) in powers of n, and then comparing the result with Eq. (D.2) we find that the terms in the cumulant expansion $B_l(\mathbf{p})$, are given for $l = 1, 2, \ldots$ by

$$B_1(\mathbf{p}) = C_p^{-1} \int d\mathbf{R}_1 \langle \mathbf{p}|\hat{u}_1(\hat{\mathbf{x}}, \mathbf{R}_1, \beta)|\mathbf{p}\rangle \equiv u_1(\mathbf{p}) \tag{D.3a}$$

$$B_2(\mathbf{p}) = u_2(\mathbf{p}) - u_1^2(\mathbf{p})/2 \tag{D.3b}$$

with

$$u_2(\mathbf{p}) \equiv \frac{C_p^{-1}}{2!} \int d\mathbf{R}_1 \int d\mathbf{R}_2 \langle \mathbf{p}|\hat{u}_2(\hat{\mathbf{x}}, \mathbf{R}_1, \mathbf{R}_2, \beta)|\mathbf{p}\rangle \tag{D.3c}$$

$$B_3(\mathbf{p}) = u_3(\mathbf{p}) - u_1(\mathbf{p})\left[u_2(\mathbf{p}) - u_1^2(\mathbf{p})/2\right] - u_1^3(\mathbf{p})/3! \tag{D.3d}$$

with

$$u_3(\mathbf{p}) \equiv \frac{C_p^{-1}}{3!} \int d\mathbf{R}_1 \int d\mathbf{R}_2 \int d\mathbf{R}_3 \langle \mathbf{p}|\hat{u}_3(\hat{\mathbf{x}}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \beta)|\mathbf{p}\rangle \tag{D.3e}$$

and so on.

Now we make use of the binary collision expansion of the cluster operators $\hat{u}_l(\hat{\mathbf{x}}, \mathbf{R}^l, \beta)$ discussed in Section 3, to write the $u_l(\mathbf{p})$ in the form of expansions in powers of the scattering length a.

For example, using Eqs. (3.7a) and (3.16) we find

$$\begin{aligned} u_1(\mathbf{p}) &= -C_p^{-1} \int d\mathbf{R}_1 \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} d\epsilon \frac{e^{\beta\epsilon}}{\left[\epsilon + E(p)\right]^2} \langle \mathbf{p}|\hat{t}(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon)|\mathbf{p}\rangle \\ &= -C_p^{-1} \frac{\Omega}{4\pi^2} \frac{a}{\hbar m} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} d\epsilon \frac{e^{\beta\epsilon}}{\left[\epsilon + E(p)\right]^2} \\ &\quad \times \left[1 + \frac{a}{\hbar}(2m\epsilon)\right]^{1/2} + O(a^2) \\ &= u_1^{(1)} + u_1^{(2)} + \cdots \end{aligned} \tag{D.4}$$

when the superscript denotes the power of a, in the scattering length expansion of $u_1(\mathbf{p})$.

Similarly the binary collision expansion of $u_2(\mathbf{p})$ has the form

$$u_2(\mathbf{p}) = C_p^{-1} \int d\mathbf{R}_1 \int d\mathbf{R}_2 \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} d\epsilon \frac{e^{\beta\epsilon}}{\left[\epsilon + E(p)\right]^2}$$

$$\times \left\{ \langle \mathbf{p}| \hat{t}\,(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon) \left[\epsilon + \hat{p}^2/2m\right]^{-1} \hat{t}\,(\hat{\mathbf{x}}, \mathbf{R}_2, \epsilon)|\mathbf{p}\rangle \right.$$

$$- \langle \mathbf{p}| \hat{t}\,(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon) \left[\epsilon + \hat{p}^2/2m\right]^{-1} \hat{t}\,(\hat{\mathbf{x}}, \mathbf{R}_2, \epsilon)$$

$$\left. \times \left[\epsilon + \hat{p}^2/2m\right]^{-1} \hat{t}\,(\hat{\mathbf{x}}, \mathbf{R}_1, \epsilon)|\mathbf{p}\rangle + \cdots \right\}$$

$$= u_{2,2}(\mathbf{p}) + u_{2,3}(\mathbf{p}) + u_{2,4}(\mathbf{p}) + \cdots \tag{D.5}$$

where the additional subscript $u_{2,i}(\mathbf{p})$ denotes the number of binary collision operators in the particular term on the right-hand side of Eq. (D.5). Since each of the $\hat{t}$ operators in the $u_{2,i}(\mathbf{p})$ has an expansion in powers of the scattering length, one can expand the $u_{2,i}(\mathbf{p})$ in powers of a as

$$u_{2,i}(\mathbf{p}) = u_{2,i}^{(i)}(\mathbf{p}) + u_{2,i}^{(i+1)} + \cdots \tag{D.6}$$

where the superscript denotes the power of a, and we use the fact that each $\hat{t}$ operator has an a expansion beginning with the first power of a. Thus

$$u_2(\mathbf{p}) = u_{2,2}^{(2)}(\mathbf{p}) + \left[u_{2,2}^{(3)}(\mathbf{p}) + u_{2,3}^{(3)}(\mathbf{p}) \right] + O(a^4) \tag{D.7}$$

One can carry out similar $\hat{t}$ and a expansion for the higher-order functions $u_l(\mathbf{p})$. When these expansions are carried out and inserted in Eq. (D.3) for the cumulant function $B_l(\mathbf{p})$, one finds that

$$B_1(\mathbf{p}) = u_1(\mathbf{p}) = u_{1,1}(\mathbf{p}) = u_{1,1}^{(1)}(\mathbf{p}) + u_{1,1}^{(2)}(\mathbf{p}) + O(a^3) \tag{D.7a}$$

$$B_2(\mathbf{p}) = u_{2,2}^{(3)}(\mathbf{p}) + u_{2,3}^{(3)}(\mathbf{p}) - u_{1,1}^{(1)}(\mathbf{p}) + u_{1,1}^{(2)}(\mathbf{p}) + O(a^4) \tag{D.7b}$$

$$B_3(\mathbf{p}) = u_{3,3}^{(4)}(\mathbf{p}) + u_{3,4}^{(4)}(\mathbf{p}) - u_{1,1}^{(1)}(\mathbf{p}) + u_{2,2}^{(3)}(\mathbf{p}) - u_{1,1}^{(2)}(\mathbf{p}) u_{2,2}^{(2)}(\mathbf{p})$$

$$- u_{1,1}^{(1)}(\mathbf{p}) u_{2,3}^{(3)}(\mathbf{p}) + \left[u_{1,1}^{(1)}(\mathbf{p}) \right]^2 u_{1,1}^{(2)}(\mathbf{p}) + O(a^5) \tag{D.7c}$$

and so on.

Here we have made use of the explicit form of the $u_{l,i}^{(j)}$ operators to remove terms that cancel in the expression for the $B_l(\mathbf{p})$. For use below, we

note that the explicit form of some of the $u_{l,i}^{(j)}$ are given by

$$u_{2,2}^{(3)}(\mathbf{p}) = C_p^{-1}(2\pi\hbar)^3\Omega\left(\frac{1}{4\pi^2\hbar m}\right)^2\frac{2a}{\hbar}(2m)^{1/2} \times \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}d\epsilon\frac{e^{\beta\epsilon}\epsilon^{1/2}}{\left[\epsilon+E(p)\right]^3} \tag{D.8a}$$

$$u_{2,3}^{(3)} = -C_p^{-1}\Omega(2\pi\hbar)^3\left(\frac{a}{4\pi^2\hbar m}\right)^3 2\pi^2 m(2m)^{1/2} \times \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}d\epsilon\frac{e^{\beta\epsilon}}{\epsilon^{1/2}\left[\epsilon+E(p)\right]^2} \tag{D8.b}$$

and

$$u_{3,3}^{(4)} = -C_p^{-1}(2\pi\hbar)^6\Omega\left(\frac{a}{4\pi^2\hbar m}\right)^3\frac{3a}{\hbar}(2m)^{1/2} \times \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}d\epsilon\frac{e^{\beta\epsilon}\epsilon^{1/2}}{\left[\epsilon+E(p)\right]^4} \tag{D.8c}$$

Explicit expressions for $u_{1,1}^{(1)}$ and $u_{1,1}^{(2)}$ were previously given in Eq. (D.4).

With these preliminary manipulations we can now outline the method of Coopersmith. The idea is that if one wants to develop an expression for Z, valid when $na\lambda^2 \gg 1$, then a term-by-term evaluation of the terms in the cumulant expansion is not sufficient since it gives a representation for Z that is only useful for $na\lambda^2 \ll 1$. Instead one must resum the terms in the expansion to obtain results that can be used when $n\lambda^2 a \gg 1$. Coopersmith has carried out a partial resummation of the series developed here. That is, he considers the sum $\sum_{l=1}^{\infty} n^l B_l(\mathbf{p})$, and for each of the $B_l(\mathbf{p})$, he includes most, but not all of the terms of order a^{l+1} or less, in the scattering length. In particular, he neglects all the terms that contain $u_{l,l+1}^{(l+1)}$, such as $u_{2,3}^{(3)}$, $u_{3,4}^{(4)}$, ... appearing in Eq. (3.7), and its generalization to higher order in l. When the terms in which $u_{l,l+1}^{(l+1)}(\mathbf{p})$ appear are neglected, and the sum over all orders in density is carried out, Coopersmith obtains the result Z_c, given by

$$Z_c = (2\pi\hbar)^{-3}\int d\mathbf{p}\, e^{-\beta p^2/2m}\exp(-\beta na\hbar^2 2\pi/m) \times \exp\left\{\frac{2(2\pi)^{1/2}na^2\hbar}{m^{1/2}\kappa}\left[(\beta\kappa^2-1)M(\beta^{1/2}\kappa)-\beta^{1/2}\kappa e^{\beta\kappa^2/2}\right]\right\} \tag{D.9}$$

with

$$\kappa = \left(\frac{p^2}{m}+\frac{4\pi na\hbar^2}{m}\right)^{1/2} \tag{D.10a}$$

and

$$M(x) = \sqrt{2} \int_0^{x/\sqrt{2}} dy\, e^{y2} \tag{D.10b}$$

The analysis of localization made by Coopersmith is based in large part on the density dependence of this quantity, Z_c. However, the difficulty with his analysis is that when the terms involving $u_{l,l+1}^{(l+1)}$ are taken into account, all the effects cancel. To see this, consider for example, $B_2(\mathbf{p})$, Eq. (C.7b). The terms included by Coopersmith are

$$\begin{aligned} B_{2,c}(\mathbf{p}) &= u_{2,2}^{(3)}(\mathbf{p}) - u_{1,1}^{(1)}(\mathbf{p}) u_{1,1}^{(2)}(\mathbf{p}) \\ &= a\left(\frac{a\hbar^2 2\pi}{m}\right)^2 \frac{(2m)^{1/2}}{\hbar} e^{\beta p^2/2m} \frac{1}{2\pi i} \\ &\quad \times \int_{\gamma - i\infty}^{\gamma + i\infty} d\epsilon\, e^{\beta\epsilon} \left[\frac{2\epsilon^{1/2}}{\left[\epsilon + E(p)\right]^3} - \frac{\beta\sqrt{\epsilon}}{\left[\epsilon + E(p)\right]^2} \right] \end{aligned} \tag{D.11}$$

However when the term $u_{2,3}^{(3)}$ is added to this one obtains

$$\begin{aligned} B_2(p) = &-a\left(\frac{a\hbar^2 2\pi}{m}\right)^2 \frac{(2m)^{1/2}}{\hbar} e^{\beta p^2/2m} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} d\epsilon \frac{d}{d\epsilon} \frac{\sqrt{\epsilon}\, e^{\beta\epsilon}}{\left[\epsilon + E(p)\right]^2} \\ &+ O(a^4) = 0 + O(a^4) \end{aligned} \tag{D.12}$$

A similar cancellation occurs at every order in the density, and the final, consistent result is

$$Z = \frac{e^{-\beta E_0}}{\lambda^3} \left[1 + O\left((na^2\lambda)^2\right) \right] \tag{D.13}$$

where $E_0 = na\hbar^2 2\pi/m$.

This is in agreement with the result obtained here by another method, cf. below Eq. (3.23), and it does not contain any features as yet useful for a description of localization.

REFERENCES

1. J. R. Dorfman and E. G. D. Cohen, *J. Math. Phys.* **8**:282 (1967).
2. Y. Pomeau and P. Resibois, *Phys. Rep.* **19C**:63 (1975).
3. J. R. Dorfman and H. van Beijeren, in *Statistical Mechanics Part B: Time Dependent Processes*, B. Berne, ed. (Plenum Press, New York, 1977).
4. K. Schwarz, *Phys. Rev. B* **21**:5125 (1980).
5. M. Goldberg and K. Watson, *Collision Theory* (Wiley, New York, 1964).
6. A. Isihara, *Statistical Physics* (Academic Press, New York), p. 369.
7. R. Zwanzig, *Phys. Rev.* **129**:486 (1963).
8. T. D. Lee and C. N. Yang, *Phys. Rev.* **113**:1165 (1959).

9. P. Resibois, *Physica* **27**:541 (1961).
10. J. M. J. van Leeuwen and A. Weijland, *Physica* **36A**:457 (1967).
11. H. Neustadter and M. Coopersmith, *Phys. Rev. Lett.* **23**:585 (1969).
12. G. Braglia and V. Dallacasa, *Phys. Rev. A* **18**:711 (1978); *Phys. Rev. A* **26**:902 (1982).
13. T. Eggarter and M. Cohen, *Phys. Rev. Lett.* **25**:807 (1970).
14. J. M. J. van Leeuwen and A. S. Reiner, *Physica* **27**:99 (1961).
15. M. H. Ernst, J. R. Dorfman, W. R. Hoegy, and J. M. J. van Leeuwen, *Physica* **45**:127 (1969).
16. R. Peierls, *Surprises in Theoretical Physics* (Princeton University Press, Princeton, New Jersey, 1979), p. 127.